Monotonicity formulas for variational problems

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1 Introduction

1.1 Monotonicity and entropy methods. This expository paper is a revision of a short talk I gave at a meeting on convexity and entropy methods at The Kavli Royal Society International Centre at Chicheley Hall during June, 2011. Most of the lectures concerned "entropy" methods for partial differential equations, which mostly mean discovering and exploiting dissipation inequalities for time-dependent PDE: see the accompanying articles and my survey [E2].

In this paper I want to advertise rather monotonicity methods for time-independent variational PDE. But let me emphasize that *these techniques are strongly related, the primary difference being that in monotonicity formulas the relevant parameter is a spatial scale r rather than time t for entropy methods.*

These approaches reflect the insight that nonlinear PDE are generally too hard to grapple with directly, and so often a good idea is to simplify by integrating out some of the variables. For monotonicity formulas we integrate various expressions involving the solution over a ball $B(x, r)$ of center x and radius r, and try to get useful differential inequalities determining how these integrals depend on the radius r. For entropy methods, we instead integrate various expressions with respect to x over all of space and try to discover useful inequalities regarding t dependence. Nevertheless, the basic technical issues are quite similar in practice. The artistry for both approaches is of course in the design of the precise expressions that we integrate.

1.2 Energy functionals and quasiconvexity. This paper concerns energy functionals of the form

$$I[\mathbf{u}] := \int_U F(D\mathbf{u})\, dx, \tag{1.1}$$

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where $\mathbf{u} : U \to \mathbb{R}^m$, $\mathbf{u} = (u^1, u^2, \dots, u^m)$, and $F : \mathbb{M}^{m\times n} \to \mathbb{R}$, $\mathbb{M}^{m\times n}$ denoting the space of real, $m \times n$ matrices. In (1.1) $D\mathbf{u}$ is the gradient matrix of $\mathbf{u}$.

A basic problem in the calculus of variations is finding minimizers (or other critical points) of such energy functions, subject to various constraints, such as given boundary conditions. That such minimization problems be well-posed requires conditions on the nonlinear term F. We say that F is *quasiconvex* provided

$$F(P)|B(0,1)| \leq \int_{B(0,1)} F(P + D\mathbf{v})\,dx$$

for all matrices $P \in \mathbb{M}^{m\times n}$ and for all C^1 functions $\mathbf{v}$ that vanish on $\partial B(0,1)$. Consult Ball [B] for the physical and mathematical importance of quasiconvexity.

1.3 Important questions. The most fundamental open problems for quasiconvex F are these:

(i) Study the regularity of minimizers, especially for the singular case that $F(P) \to \infty$ as $\det P \to 0$.

Again, see Ball [B] for a discussion of the physical relevance of such a condition in the theory of nonlinear elasticity.

(ii) Study existence, uniqueness and regularity issues for the L^2-gradient flow system:

$$u^k_t - \left(F_{p^k_\alpha}(D\mathbf{u})\right)_{x_\alpha} = 0 \qquad (k = 1, \dots, m). \tag{1.2}$$

This is the "heat equation" corresponding to (1.1).

(iii) Study existence, uniqueness and regularity questions for the corresponding "wave equation" system:

$$u^k_{tt} - \left(F_{p^k_\alpha}(D\mathbf{u})\right)_{x_\alpha} = 0 \qquad (k = 1, \dots, m). \tag{1.3}$$

Essentially nothing is known in general about the systems of PDE (1.2), (1.3) when F is merely quasiconvex, and not convex. (But see Müller–Rieger–Sverak [M-R-S]. Consult also [E-G-S] for a very special case of (1.2), corresponding to an energy density F depending only upon $\det D\mathbf{u}$.)

1.4 The Euler–Lagrange equation and its variants. To discover useful monotonicity formulas, we first proceed formally and assume that we have a solution $\mathbf{u}$ of the Euler–Lagrange system of PDE associated with the energy functional (1.1). For the case at hand, this reads

$$\left(F_{p^k_\alpha}(D\mathbf{u})\right)_{x_\alpha} = 0 \qquad (k = 1, \dots, m). \tag{1.4}$$

A key idea is to rewrite this into several more complicated, but more useful, forms. First, note that (1.4) implies

$$\left(F(D\mathbf{u})\delta_{\alpha\beta} - F_{p^k_\alpha}(D\mathbf{u})u^k_{x_\beta}\right)_{x_\alpha} = 0 \qquad (\beta = 1,\dots,n). \tag{1.5}$$

Next, multiply by x_β and sum on β:

$$\left(F(D\mathbf{u})x_\alpha - F_{p^k_\alpha}(D\mathbf{u})u^k_{x_\beta}x_\beta\right)_{x_\alpha} = nF(D\mathbf{u}) - F_{p^k_\alpha}(D\mathbf{u})u^k_{x_\alpha}. \tag{1.6}$$

Now use (1.4) in the previous formula:

$$\left(F(D\mathbf{u})x_\alpha - F_{p^k_\alpha}(D\mathbf{u})(u^k_{x_\beta}x_\beta - u^k)\right)_{x_\alpha} = nF(D\mathbf{u}). \tag{1.7}$$

It turns out that the identity (1.6) will be most immediately useful.

1.5 Target and domain variations. The foregoing formulas, while formally valid, may not make sense for nonsmooth minimizers $\mathbf{u}$. We interpret (1.4) as meaning

$$\int_U F_{p^k_\alpha}(D\mathbf{u})v^k_{x_\alpha}\,dx = 0 \tag{1.8}$$

for all smooth $\mathbf{v} = (v^1, v^2, \dots, v^m)$ that vanish near ∂U. This is the standard weak form, resulting from a standard *target variation* argument. But the integral expression in (1.8) need not be defined if F is singular. The concern is that $I[\mathbf{u}] < \infty$ need not necessarily imply $|DF(D\mathbf{u})| \in L^1_{\text{loc}}$, and this is certainly a major issue for the singular problems discussed later in Section 4. We call $\mathbf{u}$ a *critical point* provided the integral identity (1.8) makes sense and is valid for all $\mathbf{v}$ as above.

Similarly, we understand (1.5) to mean

$$\int_U \left(F(D\mathbf{u})\delta_{\alpha\beta} - F_{p^k_\alpha}(D\mathbf{u})u^k_{x_\beta}\right) w^\beta_{x_\alpha}\,dx = 0 \tag{1.9}$$

for all smooth $\mathbf{w} = (w^1, w^2, \dots, w^n)$ that vanish near ∂U. This identity is rigorously derived using a standard *domain variation* proof, and can sometimes make sense even when (1.8) does not. We call $\mathbf{u}$ a *stationary point* provided the integral identity (1.9) makes sense and is valid for all $\mathbf{w}$.

The formal, and sometimes rather intricate, calculations we provide below can be made rigorous for appropriate stationary solutions; although I do not provide the full details in this expository presentation.

2 Some first examples

In this section we discuss some selected formal derivations of monotonicity formulas for certain variational problems with convex integrands.

2.1 The 1-Laplacian PDE. For the first example we take $m = 1$ and $F(p) = |p|$. Then

$$I[u] := \int_U |Du|\,dx$$

and the formal Euler–Lagrange equation (1.4) reads

$$\left(\tfrac{u_{x_\alpha}}{|Du|}\right)_{x_\alpha} = 0. \tag{2.1}$$

The differential operator on the left is called the 1–*Laplacian*, and for expository ease we here and hereafter ignore the possibility that $|Du| = 0$. Our third basic identity (1.6) says

$$\left(|Du|x_\alpha - \tfrac{u_{x_\alpha}}{|Du|}u_{x_\beta}x_\beta\right)_{x_\alpha} = (n-1)|Du|. \tag{2.2}$$

The numerical term $n-1$ on the right is important, since it determines the scaling in r. To see this, compute

$$\begin{aligned}
\frac{d}{dr}\left(\frac{1}{r^{n-1}}\int_{B(0,r)}|Du|\,dx\right) &= \frac{1}{r^{n-1}}\int_{\partial B(0,r)}|Du|\,dS - \frac{n-1}{r^n}\int_{B(0,r)}|Du|\,dx \\
&= \frac{1}{r^{n-1}}\int_{\partial B(0,r)}|Du|\,dS - \frac{1}{r^n}\int_{\partial B(0,r)}\left(|Du|x_\alpha - \tfrac{u_{x_\alpha}}{|Du|}u_{x_\beta}x_\beta\right)\nu^\alpha\,dS \\
&= \frac{1}{r^{n-1}}\int_{\partial B(0,r)}\tfrac{u_{x_\alpha}\nu^\alpha u_{x_\beta}\nu^\beta}{|Du|}dS \geq 0,
\end{aligned}$$

where $\nu = x/r$ is the outward pointing unit normal to $\partial B(0,r)$. It follows that if $B(0,R) \subset U$ and $0 < r < R$, we have

$$\frac{1}{r^{n-1}}\int_{B(0,r)}|Du|\,dx + \int_{B(0,R)-B(0,r)}\tfrac{1}{|Du|}\tfrac{|Du\cdot x|^2}{|x|^{n+1}}\,dx = \frac{1}{R^{n-1}}\int_{B(0,R)}|Du|\,dx; \tag{2.3}$$

and hence

$$\frac{1}{r^{n-1}}\int_{B(0,r)}|Du|\,dx \leq \frac{1}{R^{n-1}}\int_{B(0,R)}|Du|\,dx. \tag{2.4}$$

The monotonicity formula (2.4) contains geometric information. To extract this, note that formally at least if u solves the 1-Laplacian PDE (2.1), then so does $v := \phi(u)$ for any function $\phi : \mathbb{R} \to \mathbb{R}$. Putting v in (2.4) and letting ϕ approximate the identity times the indicator function of an interval $[a,b] \subset \mathbb{R}$, we learn that

$$\frac{1}{r^{n-1}} \int_{B(0,r)\cap\{a\le u\le b\}} |Du|\,dx \le \frac{1}{R^{n-1}} \int_{B(0,R)\cap\{a\le u\le b\}} |Du|\,dx.$$

Dividing by $b-a$ and passing to limits, we deduce using the Coarea Formla ([E1, Section C.3]) that

$$\frac{\mathcal{H}^{n-1}(B(0,r)\cap\Gamma)}{r^{n-1}} \le \frac{\mathcal{H}^{n-1}(B(0,R)\cap\Gamma)}{R^{n-1}} \tag{2.5}$$

where Γ denotes the level surface $\{u = a\}$ and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure. This is a standard monotonicity formula for minimal hypersurfaces, valid since the 1-Laplacian PDE (2.1) implies formally that each level surface of u has zero mean curvature.

2.2 Harmonic maps into spheres. Now take $m > 1$ and consider the problem of finding a critical point of the energy functional

$$I[\mathbf{u}] := \frac{1}{2}\int_U |D\mathbf{u}|^2\,dx,$$

subject to the pointwise constraint that

$$|\mathbf{u}|^2 \equiv 1, \tag{2.6}$$

that is, that $\mathbf{u}$ take values in the unit sphere S^{m-1}. The corresponding Euler-Lagrange system of PDE is

$$-\Delta\mathbf{u} = |D\mathbf{u}|^2\mathbf{u}, \tag{2.7}$$

the term $|D\mathbf{u}|^2$ being the Lagrange multiplier for the constraint (2.6).

It is an interesting observation that although the right-hand side of (2.7) is nonzero owing to the Lagrange multiplier term, nevertheless the identity (1.5) is still valid for $F(P) = \frac{1}{2}|P|^2$. To see this, compute using (2.7) that

$$\left(\frac{1}{2}|D\mathbf{u}|^2\delta_{\alpha\beta} - u^k_{x_\alpha}u^k_{x_\beta}\right)_{x_\alpha} = -\Delta u^k u^k_{x_\beta} = |D\mathbf{u}|^2 u^k u^k_{x_\beta} = 0 \tag{2.8}$$

for $\beta = 1,\dots,n$, since (2.6) implies $u^k u^k_{x_\beta} \equiv 0$. Our third basic identity (1.6) therefore holds, and says for the case at hand that

$$\left(|D\mathbf{u}|^2 x_\alpha - 2u^k_{x_\alpha}u^k_{x_\beta}x_\beta\right)_{x_\alpha} = (n-2)|D\mathbf{u}|^2. \tag{2.9}$$

Similarly to the previous example, the numerical term $n-2$ determines the scaling for a monotonicity formula:

$$\begin{aligned}\frac{d}{dr}\left(\frac{1}{r^{n-2}}\int_{B(0,r)}|D\mathbf{u}|^2\,dx\right) &= \frac{1}{r^{n-2}}\int_{\partial B(0,r)}|D\mathbf{u}|^2\,dS - \frac{n-2}{r^{n-1}}\int_{B(0,r)}|D\mathbf{u}|^2\,dx \\ &= \frac{1}{r^{n-2}}\int_{\partial B(0,r)}|D\mathbf{u}|^2\,dS \\ &\qquad - \frac{1}{r^{n-1}}\int_{\partial B(0,r)}\left(|D\mathbf{u}|^2 x_\alpha - 2u^k_{x_\alpha}u^k_{x_\beta}x_\beta\right)\nu^\alpha\,dS \\ &= \frac{2}{r^{n-2}}\int_{\partial B(0,r)}u^k_{x_\alpha}\nu^\alpha u^k_{x_\beta}\nu^\beta dS.\end{aligned}$$

Consequently

$$\frac{1}{r^{n-2}}\int_{B(0,r)}|D\mathbf{u}|^2\,dx + 2\int_{B(0,R)-B(0,r)}\tfrac{|D\mathbf{u}\cdot x|^2}{|x|^n}\,dx = \frac{1}{R^{n-2}}\int_{B(0,R)}|D\mathbf{u}|^2\,dx,$$

and therefore

$$\frac{1}{r^{n-2}}\int_{B(0,r)}|D\mathbf{u}|^2\,dx \le \frac{1}{R^{n-2}}\int_{B(0,R)}|D\mathbf{u}|^2\,dx \tag{2.10}$$

whenever $B(0,R)\subset U$ and $0<r<R$.

Consult my old paper [E3] for the use of the monotonicity formula (2.10) to prove partial regularity of stationary harmonic maps into spheres, and see Bethuel [Be] for the generalization to stationary harmonic maps into general target manifolds.

3 A singular quasiconvex functional

We devote this section to deriving some monotonicity and almost monotonicity formulas for a model variational problem with $m=n>1$ and a singular quasiconvex energy integrand $F:\mathbb{M}^{n\times n}\to[0,\infty]$. We take

$$I[\mathbf{u}] := \int_U \tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{1}{(\det D\mathbf{u})^\gamma}\,dx, \tag{3.1}$$

where $\gamma>0$, and assume hereafter that $\mathbf{u}$ is a finite energy minimizer of the functional (3.1), relative to given, but here unspecified, boundary conditions. We suppose in particular that $\mathbf{u}\in H^1(U)$ and $(\det D\mathbf{u})^{-\gamma}\in L^1(U)$.

3.1 A monotonicity formula. For the energy functional (3.1) we have

$$F(P) = \begin{cases} \frac{1}{2}|P|^2 + \frac{1}{(\det P)^\gamma} & \text{if } \det P > 0 \\ +\infty & \text{if } \det P \leq 0; \end{cases}$$

in which case

$$F_{p^k_\alpha}(P) = p^k_\alpha - \tfrac{\gamma\,(\mathrm{cof} P)^k_\alpha}{(\det P)^{\gamma+1}},$$

when $\det P > 0$, $\mathrm{cof}P$ denoting the cofactor matrix of P. Recall for later use the identity

$$(\mathrm{cof}P)^k_\alpha p^k_\beta = \delta_{\alpha\beta} \det P. \tag{3.2}$$

Proceeding formally, we apply our third basic identity (1.6), and discover that

$$\begin{aligned} \left(F x_\alpha - \left(u^k_{x_\alpha} - \tfrac{\gamma\,(\mathrm{cof} D\mathbf{u})^k_\alpha}{(\det D\mathbf{u})^{\gamma+1}}\right) u^k_{x_\beta} x_\beta\right)_{x_\alpha} &= nF - \left(u^k_{x_\alpha} - \tfrac{\gamma\,(\mathrm{cof} D\mathbf{u})^k_\alpha}{(\det D\mathbf{u})^{\gamma+1}}\right) u^k_{x_\alpha} \\ &= nF - |D\mathbf{u}|^2 + \tfrac{\gamma n}{(\det D\mathbf{u})^\gamma} \\ &= (n-2)\left(\tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\right) + \tfrac{2(\gamma+1)}{(\det D\mathbf{u})^\gamma}. \end{aligned} \tag{3.3}$$

Therefore

$$\begin{aligned} \frac{d}{dr}\left(\frac{1}{r^{n-2}} \int_{B(0,r)} \tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\, dx\right) &= \frac{1}{r^{n-1}} \int_{B(0,r)} \tfrac{2(\gamma+1)}{(\det D\mathbf{u})^\gamma}\, dx \\ &\quad + \frac{1}{r^{n-2}} \int_{\partial B(0,r)} u^k_{x_\alpha} \nu^\alpha u^k_{x_\beta} \nu^\beta\, dS \geq 0. \end{aligned} \tag{3.4}$$

Consequently, we have the first monotonicity formula

$$\frac{1}{r^{n-2}} \int_{B(0,r)} \tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\, dx \leq \frac{1}{R^{n-2}} \int_{B(0,R)} \tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\, dx \tag{3.5}$$

for all $0 < r < R$, provided $B(0,R) \subset U$.

We can extract a bit more information by actually integrating (3.4) and keeping all the terms. To do so, we rename the radial variable s and integrate from r to R:

$$\begin{aligned} &\frac{1}{r^{n-2}} \int_{B(0,r)} \tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\, dx + \int_{B(0,R)-B(0,r)} \tfrac{|D\mathbf{u}\cdot x|^2}{|x|^n}\, dx \\ &\qquad = -\int_r^R \tfrac{1}{s^{n-1}} \int_{B(0,s)} \tfrac{2(\gamma+1)}{(\det D\mathbf{u})^\gamma}\, dx ds + \frac{1}{R^{n-2}} \int_{B(0,R)} \tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\, dx. \end{aligned}$$

We write $s^{-n+1} = -(n-2)^{-1}(s^{-n+2})'$ and integrate by parts, to derive for $n \geq 3$ the identity

$$\text{(3.6)} \quad \frac{1}{r^{n-2}} \int_{B(0,r)} \tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{n}{n-2}\tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\,dx + \int_{B(0,R)-B(0,r)} \tfrac{|D\mathbf{u}\cdot x|^2}{|x|^n} + \tfrac{2(\gamma+1)}{(n-2)|x|^{n-2}(\det D\mathbf{u})^\gamma}\,dx$$
$$= \frac{1}{R^{n-2}} \int_{B(0,R)} \tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{n}{n-2}\tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\,dx.$$

Sending $r \to 0$, we derive a bound on

$$\int_{B(0,R)} \tfrac{|D\mathbf{u}\cdot x|^2}{|x|^n} + \tfrac{1}{|x|^{n-2}(\det D\mathbf{u})^\gamma}\,dx$$

for each ball $B(0,R) \subset U$.

3.2 An almost-monotonicity formula. We can get other information by rearranging the right hand side of (3.3) differently:

$$\text{(3.7)} \quad \begin{aligned} \left(Fx_\alpha - \left(u^k_{x_\alpha} - \tfrac{\gamma\,(\mathrm{cof}D\mathbf{u})^k_\alpha}{(\det D\mathbf{u})^{\gamma+1}}\right)u^k_{x_\beta}x_\beta\right)_{x_\alpha} &= nF - \left(u^k_{x_\alpha} - \tfrac{\gamma\,(\mathrm{cof}D\mathbf{u})^k_\alpha}{(\det D\mathbf{u})^{\gamma+1}}\right)u^k_{x_\alpha} \\ &= nF - |D\mathbf{u}|^2 + \tfrac{\gamma n}{(\det D\mathbf{u})^\gamma} \\ &= n\left(\frac{1}{2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\right) - |D\mathbf{u}|^2. \end{aligned}$$

This looks good in that n now replaces $n-2$ on the right, but at the expense of the term $-|D\mathbf{u}|^2$, which has a bad sign, as we will see. Using (3.7) we calculate

$$\frac{d}{dr}\left(\frac{1}{r^n}\int_{B(0,r)} \tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\,dx\right) = \frac{1}{r^n}\int_{\partial B(0,r)} u^k_{x_\alpha}\nu^\alpha u^k_{x_\beta}\nu^\beta\,dS - \frac{1}{r^{n+1}}\int_{B(0,r)} |D\mathbf{u}|^2\,dx.$$

Relabel the radial variable s and integrate from r to R:

$$\frac{1}{r^n}\int_{B(0,r)} \tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\,dx + \int_{B(0,R)-B(0,r)} \tfrac{|D\mathbf{u}\cdot x|^2}{|x|^{n+2}}\,dx$$
$$= \int_r^R \tfrac{1}{s^{n+1}}\int_{B(0,s)} |D\mathbf{u}|^2\,dx ds + \frac{1}{R^n}\int_{B(0,R)} \tfrac{1}{2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\,dx.$$

We write $s^{-n-1} = -n^{-1}(s^{-n})'$ and integrate the first term on the left hand side, to derive the integral identity

$$\text{(3.8)} \quad \frac{1}{r^n}\int_{B(0,r)} \tfrac{2n}{n-2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\,dx = \frac{1}{R^n}\int_{B(0,R)} \tfrac{2n}{n-2}|D\mathbf{u}|^2 + \tfrac{\gamma+1}{(\det D\mathbf{u})^\gamma}\,dx$$
$$+ \int_{B(0,R)-B(0,r)} \tfrac{1}{|x|^n}\left(\tfrac{|D\mathbf{u}|^2}{n} - \tfrac{|D\mathbf{u}\cdot x|^2}{|x|^2}\right)dx.$$

However, this does not imply a true monotonicity formula, as we do not know the sign of the last term. However, we can utilize this formula to compare the L^q integrability of $|D\mathbf{u}|$ and the $L^{q/2}$ integrability of $(\det D\mathbf{u})^{-\gamma}$. For this, first let $r \to 0$ in (3.8). Then

$$\frac{\gamma+1}{(\det D\mathbf{u}(0))^{\gamma}} \leq C + C \int_{B(0,R_0)} u^k_{x_\alpha} u^k_{x_\beta} K_{\alpha\beta}(x)\, dx \tag{3.9}$$

for

$$K_{\alpha\beta}(x) := \frac{1}{|x|^n} \left(\frac{\delta_{\alpha\beta}}{n} - \frac{x_\alpha x_\beta}{|x|^2} \right) \qquad (1 \leq \alpha, \beta \leq n).$$

We observe that for each α and β, $K_{\alpha\beta}$ is a Calderon–Zygmund kernel, since

$$\int_{\partial B(0,1)} K_{\alpha\beta}\, dS = 0, \quad K_{\alpha\beta}(\lambda x) = \lambda^{-n} K_{\alpha\beta}(x).$$

Take any open subset $V \subset\subset U$ and let $R = \operatorname{dist}(V, \partial U) > 0$. We replace the center 0 by any point $y \in V$, to rewrite (3.9) in the form

$$\frac{\gamma+1}{(\det D\mathbf{u}(y))^{\gamma}} \leq C + C \int_{B(y,R)} u^k_{x_\alpha} u^k_{x_\beta} K_{\alpha\beta}(x-y)\, dx. \tag{3.10}$$

According to the Calderon–Zygmund estimates (see Stein [S]), we have for each $1 < p < \infty$ the estimate

$$||(\det D\mathbf{u})^{-\gamma}||_{L^p(V)} \leq C(|||D\mathbf{u}|^2||_{L^p(U)} + 1).$$

These formal calculations can be made rigorous, since a minimizer $\mathbf{u}$ is stationary with respect to domain variations. Then if $\mathbf{u} \in W^{1,q}(U)$ for some $2 < q < \infty$, we have

$$(\det D\mathbf{u})^{-\gamma} \in L^{q/2}_{\text{loc}}(U). \tag{3.11}$$

In other words, if we somehow know $D\mathbf{u}$ is better than square integrable, then the singular term $(\det D\mathbf{u})^{-\gamma}$ is likewise better than just integrable. This is a nontrivial deduction, with I think an interesting proof; but I do not see any immediate applications.

The foregoing is a special case of higher integrability assertions in Bauman–Owen–Phillips [B-O-P], which interested readers should read for a deeper and rigorous study of this fascinating problem. The proofs in [B-O-P] do not use monotonicity formulas, and apply to more general integrands. In fact the PDE methods of [B-O-P] make it clear that our calculations above are really just a disguised variant of standard monotonicity calculations involving the Laplacian ([E1, Section 2.2]).

3.3 Concluding comments. The foregoing attempts to find interesting monotonicity formulas for the singular quasiconvex problem (3.1) are undertaken in hopes of eventually proving partial regularity for minimizers. The methods in my paper [E4] and in subsequent work are not directly applicable, since the blow-up as $\det D\mathbf{u} \to 0$ seems preclude any direct use of minimality. My real expectation is that the monotonicity formulas above, and maybe other more sophisticated ones, will eventually provide enough extra information to fashion a partial regularity proof. But so far this is out of reach.

Also, we have thus far not used the fourth form of the Euler–Lagrange equations (1.7), which implies for any F the identity

$$(3.12)\quad \frac{1}{r^n}\int_{B(0,r)} F(D\mathbf{u})\,dx = \frac{1}{R^n}\int_{B(0,R)} F(D\mathbf{u})\,dx - \int_{B(0,R)-B(0,r)} \frac{F_{p^k_\alpha}(D\mathbf{u})x_\alpha(u^k_{x_\beta}x_\beta - u^k)}{|x|^{n+2}}\,dx.$$

Our sending $r \to 0$ gives a pointwise bound for $F(D\mathbf{u}(0))$. However the last term on the right can be extremely singular as $r \to 0$, and I do not know any interesting examples for which I can usefully estimate this expression. Knops and Stuart [K-S] have used (1.7), to prove the uniqueness of smooth solutions to the Euler–Lagrange equation for quasiconvex F, subject to linear boundary conditions.

The forthcoming paper [K-M] by Kristensen and Mingione presents some further interesting applications of monotonicity formulas to quasiconvex integrands.

References

[B] J. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Rational Mech. Analysis 63 (1976/77), 337–403.

[B-O-P] P. Bauman, N. Owen and D. Phillips, Maximal smoothness of solutions to certain Euler-Lagrange equations from nonlinear elasticity, Proc. Roy. Soc. Edinburgh Sect. A 119 (1991), 241–263.

[Be] F. Bethuel, On the singular set of stationary harmonic maps, Manuscripta Math. 78 (1993), 417–443.

[E1] L. C. Evans, *Partial Differential Equations*, Second edition, American Math. Society, 2010.

[E2] L. C. Evans, A survey of entropy methods for partial differential equations, Bulletin of the American Math Society 41 (2004), 409–438.

[E3] L. C. Evans, Partial regularity for stationary harmonic maps into spheres, Arch. Rational Mech. Analysis 116 (1991), 101–113.

[E4] L. C. Evans, Quasiconvexity and partial regularity in the calculus of variations. Arch. Rational Mech. Analysis 95 (1986), 227–252.

[E-G-S] L. C. Evans, W. Gangbo and O. Savin, Diffeomorphisms and nonlinear heat flows, SIAM J. Math. Analysis 37 (2005), 737–751.

[K-M] J. Kristensen and G. Mingione, Monotonicity formulas and regularity for minimizers of multiple integrals, to appear

[K-S] R. Knops and C. Stuart, Quasiconvexity and uniqueness of equilibrium solutions in nonlinear elasticity, Arch. Rational Mech. Analysis 86 (1984), 233249.

[M-R-S] S. Müller, M. O. Rieger and V. Sverak, Parabolic systems with nowhere smooth solutions, Arch. Rational Mech. Analysis 177 (2005), 1–20.

[S] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton U Press, 1970.