A "Lakes and Rivers" Interpretation for the Singular Limit of a Nonlinear Diffusion PDE

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Abstract

We introduce a new PDE free boundary problem that in a singular limit admits an interesting metaphoric interpretation, describing how "water" originating from "rainfall" flows along a landscape of varying height, forming "lakes" and also "rivers" and "surface flows", the latter described by transport equations along downhill gradient flows.

1 Introduction

We study in this paper a new nonlinear, multivalued PDE that admits in a singular asymptotic limit an interpretation in terms of "lakes, rivers and surface flows" over a given landscape. This effect is, we believe, mathematically interesting, although the flow phenomena we derive differ profoundly in many ways from actual water flows along geophysical landscapes. Our arguments are in part justified by mathematical rigor, in part by formal asymptotics and in part by numerical studies.

1.1 Asymptotics for a singular nonlinear PDE. Building upon the first author's thesis [D], we propose a new and multivalued equation, depending upon a small parameter $\varepsilon > 0$ that we will send to zero. Our PDE reads

$$\text{(1.1)} \qquad \begin{cases} u^\varepsilon_t \in \nabla \cdot \left(\frac{\phi(u^\varepsilon - g)}{\varepsilon} \nabla u^\varepsilon \right) + f & \text{in } \mathbb{R}^2 \times \{t > 0\} \\ u^\varepsilon = g & \text{in } \mathbb{R}^2 \times \{t = 0\}, \end{cases}$$

where ϕ denotes the multivalued Heaviside function

$$\text{(1.2)} \qquad \phi(z) := \begin{cases} 0 & z < 0 \\ [0,1] & z = 0 \\ 1 & z > 0. \end{cases}$$

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Part of our interest in this problem is that the limit $u^0 := \lim_{\varepsilon\to 0} u^\varepsilon$ behaves extremely simply. Indeed, as we will see, our PDE forces

$$u^0 \geq g, \quad |\nabla u^0| = 0 \ \text{ on } \{u^0 > g\}.$$

We may informally interpret each component of $\{u^0 > g\}$ as a "lake" on which the height u^0 of the water is constant. In particular, at each moment of time the boundary of each lake is a level set of the altitude g, although this boundary will change as the lake "fills up". More interestingly, a lake can also overflow through a mountain pass at a saddle point of g; in which case the excess water flows downhill along a one-dimensional "river", possibly to fill up another lake at a lower elevation.

So our scenario is mostly quite simple: rain fills up lakes, which may then overflow, passing the excess water through rivers to lakes at lower elevations. But there are hidden subtleties. Think of a river leaving an overflowing uphill lake which subsequently flows downhill along a ridge line into another saddle point. What happens then? Does the river split? And if so, how is its mass distributed into individual daughter rivers flowing further downhill? Answering such questions depends upon detailed understanding of the precise asymptotic behaviour of the solutions u^ε of the PDE (1.1) for our multivalued nonlinearity (1.2). We will see in this paper that such issues can be very tricky.

1.2 Apologia. Let us pause here to comment explicitly that we are not proposing the singular limit of our nonlinear PDE (1.1)–(1.2) as a realistic model of actual water flow over geophysical landscapes. The inadequacies for physical modeling are many, and some of the rigorous deductions from our PDE are physically absurd: for example, our "rivers" cannot merge and, even worse, can run along ridge lines.

We are motivated rather by theoretical considerations, most notably searching for simpler structures appearing in asymptotic limits as $\varepsilon \to 0$ and then perturbing off these simpler structures to understand actual solutions for small $\varepsilon > 0$. *Our "lakes and rivers" interpretation is therefore merely a metaphor*, but an extremely compelling metaphor, as we will see. The reader should continually be aware that although we afterwards mostly drop the quotation marks around the words "water", "lakes", "rivers", etc. the mathematical entities so called should never be confused with actual water, lakes, rivers, etc. Indeed, our experience has shown that the foregoing "physical" interpretation of the $\varepsilon \to 0$ limit is so overwhelmingly intuitive that it actually interferes with careful mathematical analysis: many "physically obvious" statements are either quite hard to prove or are false.

Finally, we note that the asymptotic mathematical phenomena identified here, especially the advent of one-dimensional, nonlocal structures that "carry mass", are strongly connected with related effects in optimal transport theory (discussed for instance in Villani [V]).

1.3 Flowing sand. This paper is an outgrowth of the second author's previous paper [A-E-W] (written with G. Aronsson and Y. Wu) and the first author's PhD thesis [D] on fast diffusion PDE interpretations of "sandpile" dynamics. In these papers $u = u(x,t)$ denotes the height of the sandpile at $x \in \mathbb{R}^2$ and time $t \geq 0$. We assume the stability condition that always

$$|\nabla u| \leq 1, \tag{1.3}$$

where $\nabla u = \nabla_x u$ is the gradient in the spatial variable $x = (x_1, x_2)$.

Assuming that sand is added to the pile at rate $f = f(x,t) \geq 0$, we proposed in the paper [A-E-W] to approximate the height u by the solution u^p of the fast/slow diffusion PDE

$$u_t^p - \nabla \cdot (|\nabla u^p|^{p-2} \nabla u^p) = f \qquad \text{in } \mathbb{R}^2 \times [0, \infty) \tag{1.4}$$

with zero initial conditions, for $p \gg 1$. Indeed, [A-E-W] proves that as $p \to \infty$, we have $u^p \to u$, the limit satisfying the slope constraint (1.3) and the dynamics

$$f - u_t \in \partial I_\infty[u] \tag{1.5}$$

for the convex function

$$I_\infty[v] := \begin{cases} 0 & \text{if } v \in L^2(\mathbb{R}^2),\ |\nabla v| \leq 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

The point is that (1.5) is a rather simple, if nonlocal, evolution modeling the instantaneous rearrangement of the new sand added to the pile. These dynamics are consistent with an earlier sand-cone model of Aronsson [A], admit neat interpretations in terms of Monge–Kantorovich optional mass transfer [E], arise from discrete stochastic models [E-R], and—as discovered by Prigozhin and Zaltzman [P-Z]—appear as asymptotic limits of so-called BCRE (Bouchaud, Cates, Ravi Prakash and Edwards) type models in the physics literature. The PhD thesis [B] of J. Bang extends these models to flows of sand on steep landscapes and provides further motivation for the following.

Prigozhin in his earlier paper [P] independently derived the dynamics (1.5) and furthermore extended the interpretation to model flowing sand over a given landscape of height $g = g(x) \geq 0$. And, most relevantly for this paper, Prigozhin proposed also changing the stability condition (1.3) to read

$$|\nabla u| \leq \gamma \tag{1.6}$$

and then sending $\gamma \to 0$. This amounts to changing the angle of repose to zero, in which case we reinterpret u as the height of a mass of not sand, but rather "water", originating from "rainfall" of rate f and flowing across the topography determined by g.

Our PDE (1.1), with the Heaviside function (1.2), is suggested by Prigozhin's construction, but is much simpler. See also [D] for a full analysis of our "water" flow problem in one space dimension. Our two-dimensional situation is much more complicated.

We are extremely grateful to Professor John Neu for providing us with an extensive asymptotic analysis of a related model problem, for which instead of (1.2) we take $\phi(z) = z^+$. We hope to return to this alternative model in a future paper.

2 Weak solutions

In this and the next section we discuss in detail the time evolution (1.1). It will be convenient first to recast our PDE into the form

$$\text{(2.1)} \qquad \begin{cases} u_t^\varepsilon - \nabla \cdot (\frac{1}{\varepsilon}\nabla(u^\varepsilon - g)^+ + a^\varepsilon \nabla g) = f & \text{in } \mathbb{R}^2 \times \{t > 0\} \\ u^\varepsilon = g & \text{on } \mathbb{R}^2 \times \{t = 0\} \end{cases}$$

for some measurable function $a^\varepsilon = a^\varepsilon(x,t)$ satisfying

$$\text{(2.2)} \qquad a^\varepsilon \in \frac{\phi(u^\varepsilon - g)}{\varepsilon}.$$

In particular,

$$\text{(2.3)} \qquad 0 \leq a^\varepsilon \leq \frac{1}{\varepsilon}.$$

We henceforth assume that the landscape height function $g : \mathbb{R}^2 \to \mathbb{R}$ is smooth and nonnegative, with compact support. We also assume that g restricted to $\{g > 0\}$ is a Morse function; that is, g has precisely N critical points $\{x_1, \ldots, x_N\}$ within $\{g > 0\}$, and at these points $\det \nabla^2 g(x_k) \neq 0$ for $k = 1, \ldots, N$. Hence each critical point of the elevation function represents a simple peak, pit or pass. We will further suppose that $\Delta g(x_k) \neq 0$ for $k = 1, \ldots, N$, since we will see later that the sign of Δg at a saddle point strongly effects the asymptotics as $\varepsilon \to 0$.

Considering the source function $f : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$, we assume that $f \geq 0$ and that $\operatorname{spt} f \subset (\operatorname{spt} g)^0 \times (0, T)$.

2.1 Elementary estimates, existence of weak solution. We call the pair $(u^\varepsilon, a^\varepsilon)$ a *weak solution* of (2.1) provided (2.2) holds, $u^\varepsilon, a^\varepsilon \in L^\infty$, $\nabla(u^\varepsilon - g)^+ \in L^2_{loc}$, and

$$\text{(2.4)} \qquad \int_0^T \int_{\mathbb{R}^2} -u^\varepsilon \psi_t + \left(\frac{1}{\varepsilon}\nabla(u^\varepsilon - g)^+ + a^\varepsilon \nabla g\right) \cdot \nabla\psi \, dxdt = \int_0^T \int_{\mathbb{R}^2} f\psi \, dxdt + \int_{\mathbb{R}^2} g\psi(\cdot, 0)\, dx$$

for all smooth functions ψ with compact support within $\mathbb{R}^2 \times [0, T)$.

Theorem 2.1 *A weak solution $(u^\varepsilon, a^\varepsilon)$ exists. Furthermore*

$$u^\varepsilon \geq g, \tag{2.5}$$

and we have the estimates

$$\max_{\mathbb{R}^2\times[0,T]} |u^\varepsilon| \leq C, \tag{2.6}$$

$$\int_0^T \int_{\{u^\varepsilon > g\}} |\nabla u^\varepsilon|^2 \, dxdt \leq C\varepsilon \tag{2.7}$$

for some constant C independent of ε.

Proof. 1. Define for $\delta > 0$ a smooth, nondecreasing function ϕ_δ such that

$$\phi_\delta(z) = 0 \text{ if } z \leq 0; \quad \phi_\delta(z) > 0 \text{ if } z > 0; \quad \phi_\delta(z) = 1 \text{ if } z \geq \delta. \tag{2.8}$$

Then there exists a unique, smooth solution $u^{\varepsilon,\delta}$ of

$$\begin{cases} u_t^{\varepsilon,\delta} - \nabla \cdot \left(\frac{\phi_\delta(u^{\varepsilon,\delta}-g)}{\varepsilon} \nabla u^{\varepsilon,\delta}\right) - \delta \Delta u^{\varepsilon,\delta} = f & \text{in } \mathbb{R}^2 \times [0,\infty) \\ \qquad\qquad u^\varepsilon = g & \text{in } \mathbb{R}^2 \times \{t = 0\}, \end{cases} \tag{2.9}$$

with $\lim_{|x|\to\infty} u^{\varepsilon,\delta} = 0$. Multiplying by $u^{\varepsilon,\delta}$ and integrating by parts, we derive the estimate

$$\sup_{0\leq t\leq T} \int_{\mathbb{R}^2} (u^{\varepsilon,\delta})^2 \, dx + \int_0^T \int_{\mathbb{R}^2} \left(\frac{\phi_\delta(u^{\varepsilon,\delta} - g)}{\varepsilon} + \delta\right) |\nabla u^{\varepsilon,\delta}|^2 \, dxdt \leq C, \tag{2.10}$$

for a constant $C = C(T)$ independent of ε, δ. The maximum principle provides also the bound

$$\max_{\mathbb{R}^2\times[0,T]} |u^{\varepsilon,\delta}| \leq C. \tag{2.11}$$

Passing as necessary to a subsequence, we may assume $u^{\varepsilon,\delta} \rightharpoonup u^\varepsilon$ and $\nabla u^{\varepsilon,\delta} \rightharpoonup \nabla u^\varepsilon$ weakly in L^2 as $\delta \to 0$, where u^ε satisfies (2.6).

2. Select $\eta : \mathbb{R} \to \mathbb{R}$ so that η is convex, $\eta \geq 0$, $\eta' \leq 0$, and $\eta(z) > 0$ if $z < 0$; $\eta(z) = 0$ if $z \geq 0$. Then

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} \eta(u^{\varepsilon,\delta} - g)\, dx &= \int_{\mathbb{R}^2} \eta'(u^{\varepsilon,\delta} - g) u_t^{\varepsilon,\delta} \, dx \\ &= \int_{\mathbb{R}^2} -\eta''(\nabla u^{\varepsilon,\delta} - \nabla g) \cdot \nabla u^{\varepsilon,\delta} \left(\frac{\phi_\delta}{\varepsilon} + \delta\right) + \eta' f \, dx \\ &\leq C\delta \int_{\mathbb{R}^2} |\nabla u^{\varepsilon,\delta}||\nabla g| \, dx, \end{aligned}$$

since $\eta''(u^{\varepsilon,\delta}-g)\phi_\delta(u^{\varepsilon,\delta}-g)=0$ and $\eta' f\leq 0$. Consequently

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}^2}\eta(u^{\varepsilon,\delta}-g)\,dx\leq C\delta,$$

and so

$$\int_0^T\int_{\mathbb{R}^2}\eta(u^\varepsilon-g)\,dxdt\leq\liminf_{\delta\to 0}\int_0^T\int_{\mathbb{R}^2}\eta(u^{\varepsilon,\delta}-g)\,dxdt=0. \tag{2.12}$$

It follows that $u^\varepsilon\geq g$.

3. Next, fix $\sigma\geq\delta>0$ and select a smooth function $\Psi=\Psi_\sigma:\mathbb{R}\to\mathbb{R}$ so that

$$\Psi\geq 0,\quad \Psi(z)=0\ \text{ for } z\leq\sigma,\quad \Psi(z),\Psi'(z)>0\ \text{ for } z>\sigma. \tag{2.13}$$

Put $w^{\varepsilon,\delta}:=\Psi(u^{\varepsilon,\delta}-g)$. Then estimate (2.10) implies

$$\int_0^T\int_{\mathbb{R}^2}|\nabla w^{\varepsilon,\delta}|^2\,dxdt\leq C,$$

the constant independent of δ. Furthermore the PDE (2.9) implies that if ψ is smooth,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^2}w_t^{\varepsilon,\delta}\psi\,dxdt &= \int_0^T\int_{\mathbb{R}^2}\Psi'(u^{\varepsilon,\delta}-g)u_t^{\varepsilon,\delta}\psi\,dxdt\\ &= \int_0^T\int_{\mathbb{R}^2}-\nabla(\Psi'\psi)\cdot\nabla u^{\varepsilon,\delta}\left(\frac{\phi_\delta}{\varepsilon}+\delta\right)+f\Psi'\psi\,dxdt\\ &\leq C(\|\psi\|_{L^\infty}+\|\nabla\psi\|_{L^2})\\ &\leq C\|\psi\|_{W^{1,p}}\end{aligned}$$

for $p>2$. Hence $\{w_t^{\varepsilon,\delta}\}$ is bounded in $W^{-1,q}$, $\frac{1}{p}+\frac{1}{q}=1$; and consequently $w^{\varepsilon,\delta}$ is precompact in L^2. Therefore, passing to a subsequence, we may suppose $w^{\varepsilon,\delta}\to w^\varepsilon$ in L^2 and almost everywhere, as $\delta\to 0$.

4. We now claim that

$$w^\varepsilon=\Psi(u^\varepsilon-g). \tag{2.14}$$

This holds since

$$0\leq\int_0^T\int_{\mathbb{R}^2}(\Psi(u^{\varepsilon,\delta}-g)-\Psi(v))(u^{\varepsilon,\delta}-g-v)\,dx,$$

and then

$$0\leq\int_0^T\int_{\mathbb{R}^2}(w^\varepsilon-\Psi(v))(u^\varepsilon-g-v)\,dx.$$

Put $v = u^\varepsilon - g - \lambda z$ and let $\lambda \to 0^+$, to deduce

$$0 \le \int_0^T \int_{\mathbb{R}^2} (w^\varepsilon - \Psi(u^\varepsilon - g)) z \, dx$$

for all smooth functions z. This implies (2.14).

We further assert that $u^{\varepsilon,\delta} \to u^\varepsilon$ a.e. on $\{u^\varepsilon > g\}$. This follows since we have shown $\Psi(u^{\varepsilon,\delta} - g) \to \Psi(u^\varepsilon - g)$ a.e. for each function Ψ satisfying (2.13). We claim further that $u^{\varepsilon,\delta} \to u^\varepsilon = g$ a.e. on $\{u^\varepsilon = g\}$. This is so since (2.12) implies

$$\lim_{\delta \to 0} \int_0^T \int_{\{u^\varepsilon = g\}} (u^\varepsilon - u^{\varepsilon,\delta})^+ \, dxdt = 0,$$

Hence, again passing as needed to a subsequence, we have

$$u^{\varepsilon,\delta} \to u^\varepsilon \quad \text{a.e.} \tag{2.15}$$

5. We may further assume

$$\phi_\delta(u^{\varepsilon,\delta} - g) \rightharpoonup \alpha_\varepsilon \quad \text{weakly} * \text{ in } L^\infty.$$

We claim

$$\alpha_\varepsilon \in \phi(u^\varepsilon - g).$$

As $0 \le \alpha_\varepsilon \le 1$ and $u^\varepsilon \ge g$, it is enough to show $\alpha_\varepsilon = 1$ a.e. on $\{u^\varepsilon > g\}$. But this follows from (2.15).

6. Finally, define $\Phi_\delta(z) := \int_0^z \phi_\delta(s) \, ds$, and then rewrite the PDE (2.9) as

$$u_t^{\varepsilon,\delta} = \frac{1}{\varepsilon} \Delta \Phi_\delta(u^{\varepsilon,\delta} - g) + \frac{1}{\varepsilon} \nabla \cdot (\phi_\delta(u^{\varepsilon,\delta} - g) \nabla g) + \delta \Delta u^{\varepsilon,\delta} + f.$$

Passing to limits in the distribution sense, we find that

$$u_t^\varepsilon = \frac{1}{\varepsilon} \Delta (u^\varepsilon - g)^+ + \frac{1}{\varepsilon} \nabla \cdot (\alpha_\varepsilon \nabla g) + f$$

and consequently that u^ε and $a^\varepsilon = \frac{\alpha^e}{\varepsilon}$ give a weak solution of (2.1). Estimate (2.7) follows from (2.10). □

2.2 Contraction estimates. We show next that if two rainfall patterns are close in the L^1-norm, then the corresponding distributions of surface water are close.

Theorem 2.2 *Let $(u^\varepsilon, a^\varepsilon)$ and $(\hat{u}^\varepsilon, \hat{a}^\varepsilon)$ be two weak solutions constructed as in the previous proof, with rainfall terms $f, \hat{f}$. Then we have the L^1–contraction inequality*

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^2} |u^\varepsilon - \hat{u}^\varepsilon| \, dx \le \int_0^T \int_{\mathbb{R}^2} |f - \hat{f}| \, dxdt. \tag{2.16}$$

Proof. 1. Define $\Phi_\delta'(z) = \phi_\delta(z) + \varepsilon\delta$; so that, according to (2.8), Φ_δ is strictly increasing. Take η to be a smooth, convex function with $|\eta'| \leq 1$. Then

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} \eta(u^{\varepsilon,\delta} - \hat{u}^{\varepsilon,\delta})\,dx &= \int_{\mathbb{R}^2} \eta'(u^{\varepsilon,\delta} - \hat{u}^{\varepsilon,\delta})(u_t^{\varepsilon,\delta} - \hat{u}_t^{\varepsilon,\delta})\,dx \\
&\leq \int_{\mathbb{R}^2} \frac{1}{\varepsilon}\eta'(u^{\varepsilon,\delta} - \hat{u}^{\varepsilon,\delta})\nabla\cdot[(\phi_\delta(u^{\varepsilon,\delta} - g) + \varepsilon\delta)\nabla u^{\varepsilon,\delta} \\
&\qquad - (\phi_\delta(\hat{u}^{\varepsilon,\delta} - g) + \varepsilon\delta)\nabla\hat{u}^{\varepsilon,\delta}] + \eta'(f - \hat{f})\,dx \\
&= \int_{\mathbb{R}^2} \frac{1}{\varepsilon}\eta'(u^{\varepsilon,\delta} - \hat{u}^{\varepsilon,\delta})\nabla\cdot(\nabla\Phi_\delta(u^{\varepsilon,\delta} - g) - \nabla\Phi_\delta(\hat{u}^{\varepsilon,\delta} - g)) \\
&\qquad + \frac{1}{\varepsilon}\eta'(u^{\varepsilon,\delta} - \hat{u}^{\varepsilon,\delta})\nabla\cdot[(\phi_\delta(u^{\varepsilon,\delta} - g) - \phi_\delta(\hat{u}^{\varepsilon,\delta} - g))\nabla g] \\
&\qquad\qquad + \eta'(f - \hat{f})\,dx \\
&=: A + B + C.
\end{aligned}$$

2. We select a sequence $\eta = \eta_k$ such that $\eta_k(z) \to |z|$, $\eta_k'(z) \to \operatorname{sgn}(z)$ pointwise, with $|\eta_k'| \leq 1$. Then

$$\begin{aligned}
A &= \frac{1}{\varepsilon}\int_{\mathbb{R}^2} \eta_k'(\Phi_\delta(u^{\varepsilon,\delta} - g) - \Phi_\delta(\hat{u}^{\varepsilon,\delta} - g))\nabla\cdot(\nabla\Phi_\delta(u^{\varepsilon,\delta} - g) - \nabla\Phi_\delta(\hat{u}^{\varepsilon,\delta} - g))\,dx + D \\
&= -\frac{1}{\varepsilon}\int_{\mathbb{R}^2} \eta_k''(\,\cdot\,)|\nabla\Phi_\delta(u^{\varepsilon,\delta} - g) - \nabla\Phi_\delta(\hat{u}^{\varepsilon,\delta} - g)|^2\,dx + D \leq D,
\end{aligned}$$

for the term

$$\begin{aligned}
D := \frac{1}{\varepsilon}\int_{\mathbb{R}^2} &[\eta_k'((u^{\varepsilon,\delta} - g) - (\hat{u}^{\varepsilon,\delta} - g)) - \eta_k'(\Phi_\delta(u^{\varepsilon,\delta} - g) - \Phi_\delta(\hat{u}^{\varepsilon,\delta} - g))] \\
&\qquad (\Delta\Phi_\delta(u^{\varepsilon,\delta} - g) - \Delta\Phi_\delta(\hat{u}^{\varepsilon,\delta} - g))\,dx.
\end{aligned}$$

We assert that $D \to 0$ as $k \to \infty$, with $\delta > 0, \varepsilon > 0$ fixed. Indeed, since $\eta_k'(z) \to \operatorname{sgn}(z)$ and Φ_δ is strictly increasing, the integrand goes to zero pointwise. Furthermore the integrand is estimated by $\frac{2}{\varepsilon}|\Delta\Phi_\delta(u^{\varepsilon,\delta} - g) - \Delta\Phi_\delta(\hat{u}^{\varepsilon,\delta} - g)|$, which is integrable; and so the Dominated Convergence Theorem applies.

3. Next, define $\psi_\delta(z) := \phi_\delta(z) + \delta z$ and note ψ_δ is strictly increasing. Then

$$\begin{aligned}
B &= \frac{1}{\varepsilon}\int_{\mathbb{R}^2} \eta_k'(u^{\varepsilon,\delta} - \hat{u}^{\varepsilon,\delta})\nabla\cdot[(\phi_\delta(u^{\varepsilon,\delta} - g) - \phi_\delta(\hat{u}^{\varepsilon,\delta} - g))\nabla g]\,dx \\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^2} \eta_k'(\psi_\delta(u^{\varepsilon,\delta} - g) - \psi_\delta(\hat{u}^{\varepsilon,\delta} - g))\nabla\cdot[(\psi_\delta(u^{\varepsilon,\delta} - g) - \psi_\delta(\hat{u}^{\varepsilon,\delta} - g))\nabla g]\,dx \qquad (2.17) \\
&\qquad\qquad + E,
\end{aligned}$$

where

$$E := \frac{1}{\varepsilon}\int_{\mathbb{R}^2}[\eta_k'(u^{\varepsilon,\delta}-\hat{u}^{\varepsilon,\delta})-\eta_k'(\psi_\delta(u^{\varepsilon,\delta}-g)-\psi_\delta(\hat{u}^{\varepsilon,\delta}-g))]$$
$$\nabla\cdot[(\psi_\delta(u^{\varepsilon,\delta}-g)-\psi_\delta(\hat{u}^{\varepsilon,\delta}-g))\nabla g]$$
$$-\delta\eta_k'(u^{\varepsilon,\delta}-\hat{u}^{\varepsilon,\delta})\nabla\cdot[(u^{\varepsilon,\delta}-\hat{u}^{\varepsilon,\delta})\nabla g]\,dx.$$

Now the first term on the right side of (2.17) equals

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\eta_k'(\psi_\delta(u^{\varepsilon,\delta}-g)-\psi_\delta(\hat{u}^{\varepsilon,\delta}-g))(\psi_\delta(u^{\varepsilon,\delta}-g)-\psi_\delta(\hat{u}^{\varepsilon,\delta}-g))\Delta g\,dx\\
&\qquad+\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\nabla[\eta_k(\psi_\delta(u^{\varepsilon,\delta}-g)-\psi_\delta(\hat{u}^{\varepsilon,\delta}-g))]\cdot\nabla g\,dx\\
&=\frac{1}{\varepsilon}\int_{\mathbb{R}^2}[\eta_k'(\psi_\delta(u^{\varepsilon,\delta}-g)-\psi_\delta(\hat{u}^{\varepsilon,\delta}-g))(\psi_\delta(u^{\varepsilon,\delta}-g)-\psi_\delta(\hat{u}^{\varepsilon,\delta}-g))\\
&\qquad-\eta_k(\psi_\delta(u^{\varepsilon,\delta}-g)-\psi_\delta(\hat{u}^{\varepsilon,\delta}-g))]\Delta g\,dx.
\end{aligned}$$

This expression goes to zero as $k\to\infty$, since integrand has compact support and goes boundedly, pointwise to zero.

Considering next the expression E defined above, we observe that the first term goes to zero, since $\eta_k'(z)\to\operatorname{sgn}(z)$ and ψ_δ is strictly increasing. The second term is estimated by

$$\frac{C\delta}{\varepsilon}\int_{\{\operatorname{spt} g\}}|\nabla u^{\varepsilon,\delta}|+|\nabla\hat{u}^{\varepsilon,\delta}|+1\,dx.$$

4. Finally, since $|\eta_k'|\le 1$, we have

$$|C|\le\int_{\mathbb{R}^2}|f-\hat{f}|\,dx.$$

We finish the proof by integrating in time the expression derived in Step 1, recalling the estimates above, sending $k\to\infty$, and then letting $\delta\to 0$. □

3 Lakes filling up

In this section we look more closely at how our model describes "lakes filling up", both for $\varepsilon>0$ and for the limit $\varepsilon\to 0$. We expect that our solution u^ε of (2.1) subdivides $\mathbb{R}^2\times[0,T]$ into the two regions

(i) $\{u^\varepsilon>g\}$, where the "water is above the landscape" and

$$\text{(ii)} \quad \{u^\varepsilon = g\}, \text{ where the "water is not above the landscape".}$$

However there can exist a downhill "surface flow" of water within region (ii). This can arise either from "rainfall" from the source term f or "overflow" of uphill lakes. We discuss the first effect in the next subsection, and the second in Section 4.

3.1 Surface flows. To understand our claim about surface flow within the region (ii) observe that, formally at least, our PDE (2.1) reads

$$-\nabla \cdot (a \nabla g) = f \quad \text{in } \{u^\varepsilon = g\} \tag{3.1}$$

where

$$a = a^\epsilon \in \frac{\phi(u^\varepsilon - g)}{\varepsilon}. \tag{3.2}$$

Introduce the *downhill gradient flow*

$$\dot{\boldsymbol{\gamma}}(\tau) = -\nabla g(\boldsymbol{\gamma}(\tau)), \tag{3.3}$$

for $\dot{} = \frac{d}{d\tau}$. Then we can regard (3.1) as an ODE, namely

$$\dot{a} - a\Delta g = f, \tag{3.4}$$

the terms a, g, f evaluated at $x = \boldsymbol{\gamma}(\tau)$. The term a represents the *surface flow intensity.*

Consider now the case that x_0 is a isolated strict local minimum point, surrounded by its basin of attraction B under the flow (3.3), as drawn in Figure 1. Solving (3.4), we find

$$a(\boldsymbol{\gamma}(\tau), t) = \int_{-\infty}^{\tau} f(\boldsymbol{\gamma}(s), t) e^{\int_s^\tau \Delta g(\boldsymbol{\gamma}(r)) dr} ds \qquad (\tau \in \mathbb{R}). \tag{3.5}$$

The foregoing construction of the surface flow intensity is consistent with (3.2) in the region $\{u^\varepsilon = g\}$ provided $0 \leq a(\boldsymbol{\gamma}(\tau), t) \leq \frac{1}{\varepsilon}$. However if for some τ_0, we have

$$a(\tau_0) = \frac{1}{\varepsilon}, \quad \dot{a}(\tau_0) > 0 \tag{3.6}$$

our condition (3.2) would be violated for $\tau > \tau_0$. The resolution is that the trajectory $\boldsymbol{\gamma}(\cdot)$ has in fact entered region (i), where $u^\varepsilon > g$, at or before τ_0. In other words, if the surface flow intensity a exceeds $\frac{1}{\varepsilon}$, then the water level rises above the landscape height. Near the minimum point x_0, we interpret this rising of the water as the genesis of a lake, which should persist in the limit $\varepsilon \to 0$. Since we are assuming $\Delta g(x_0) > 0$ at the minimum point x_0, formula (3.5) shows that we will definitely reach the critical situation (3.6) (unless the total

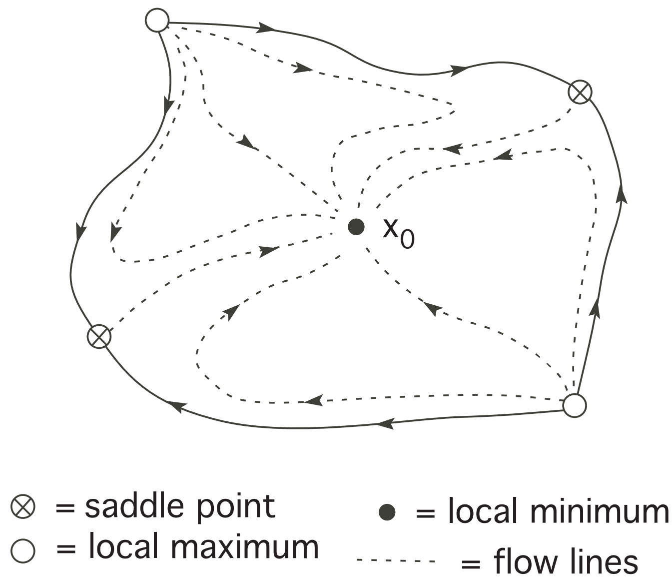

Figure 1: Surface flow within the basin of a local minimum

rainfall along the curve is $f \equiv 0$). *We have therefore proved that "if it rains, a lake must form".*

3.2 Lakes. Define now $L^\varepsilon(t)$ to be the connected component of $\{u^\varepsilon(\cdot,t) > g\}$ containing the minimum point x_0. According to (3.2) we have $a^\varepsilon \equiv \frac{1}{\varepsilon}$ on $L^\varepsilon(t)$ and consequently estimate (2.7) implies

$$\int_0^T \int_{L^\varepsilon(t)} |\nabla u^\varepsilon|^2 \, dxdt \le C\varepsilon.$$

This bound suggests that if $L^\varepsilon \to L^0(t)$, $u^\varepsilon \to u^0$ in some appropriate sense as $\varepsilon \to 0$, then $\nabla u^0 = 0$ a.e. on $L^0(t)$; and consequently if $L^0(t)$ is connected, we have $u^0(x,t) = c(t)$ for $x \in L^0(t)$. We regard $L^0(t)$ as the limiting lake, of height $c(t)$ at time t. Since $u^0 = g$ on $\partial L^0(t)$, we deduce that

$$L^0(t) = \{x \text{ near } x_0 \mid g(x) \le c(t)\}.$$

So the behavior of u^0 and $L^0(t)$ is completely trivial within the basin B:

$$u^0(x,t) = \begin{cases} c(t) & x \in L^0(t) \\ g(x) & x \in B - L^0(t). \end{cases} \tag{3.7}$$

Furthermore, at each time t

$$\int_B f(x,t)\,dx = \frac{d}{dt}\int_{L^0(t)} c(t) - g(x)\,dx = \dot{c}(t)|L^0(t)|, \tag{3.8}$$

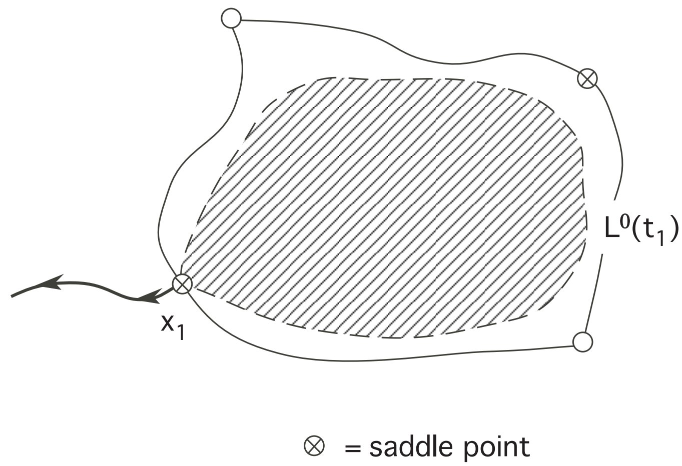

Figure 2: An overflowing lake creating a "river"

since the left-hand side of (3.8) represents the total rainfall within the basin (which instantly flows into the lake $L^0(t)$) and the right-hand side is the rate of change of the volume of water in the lake.

So the limiting lake $L^0(t)$ and height function u^0 are completely and simply characterized by (3.7), (3.8), until the first time $t^* > 0$ that $\partial L^0(t^*)$ hits the boundary of the basin B, at the saddle point with the lowest elevation. At times $t \geq t^*$, the lake $L^0(t) \equiv L^0(t^*)$ cannot grow any more, and the excess water from further rainfall within B overflows through the saddle point. See Figure 2.

3.3 Asymptotics. We propose next to analyze the behavior of u^ε and $L^\varepsilon(t)$ $(0 < t < t^*)$ as perturbations of the very simple u^0 and $L^0(t)$ introduced in the previous subsection. For this we assume within $L^\varepsilon(t)$ the ansatz

$$u^\varepsilon = u^0 + \varepsilon w^\varepsilon = c(t) + \varepsilon w^\varepsilon,$$

and try to deduce the asymptotics of w^ε as $\varepsilon \to 0$.

a. PDE for $\mathbf{w^\epsilon, w^0}$. Since $u^\varepsilon_t - \frac{1}{\varepsilon}\Delta u^\varepsilon = f$ within $L^\varepsilon(t)$, we find

$$\dot{c} + \varepsilon w^\varepsilon_t - \Delta w^\varepsilon = f \quad \text{in } L^\varepsilon(t). \tag{3.9}$$

Therefore if $w^\varepsilon \to w^0$, we expect w^0 to solve the PDE

$$-\Delta w^0 = f - \dot{c} \quad \text{in } L^0(t). \tag{3.10}$$

b. Boundary conditions for $\mathbf{w^0}$. Owing to (3.1), we have

$$\int_{B-L^0(t)} f\,dx = \int_{\partial L^0(t)} a\frac{\partial g}{\partial \nu}\,ds = \int_{\partial L^0(t)} a|\nabla g|\,ds \tag{3.11}$$

since $\frac{\partial g}{\partial \nu} = 0$ on ∂B and $\nu = \frac{\nabla g}{|\nabla g|}$ is the outward pointing unit normal to $\partial L^0(t)$. The expression on the left is the total rainfall within the basin, but outside the lake. Anticipating our later discussion of boundary conditions for w^ε, in particular formula (3.15) below, we introduce the *inflow boundary conditions*

$$\frac{\partial w^0}{\partial \nu} = a\frac{\partial g}{\partial \nu} = a|\nabla g| \quad \text{on } \partial L^0(t), \tag{3.12}$$

where a is computed by solving the flow equations (3.3), (3.4) in the region $B - L^0(t)$. According to (3.8) and (3.11), we thus have the compatibility condition

$$\int_{L^0(t)} f - \dot{c}\, dx = -\int_{B-L^0(t)} f\, dx = \int_{\partial L^0(t)} a|\nabla g|\, ds;$$

and consequently the PDE (3.10), (3.12), with the normalization

$$\int_{L^0(t)} w^0\, dx = 0, \tag{3.13}$$

is uniquely solvable for w^0.

c. Boundary conditions for $\mathbf{w}^\epsilon$. We need next to discover the appropriate boundary conditions for w^ε. We firstly have

$$w^\varepsilon = \frac{g - c}{\varepsilon} \quad \text{on } \partial L^\varepsilon(t), \tag{3.14}$$

since $u^\varepsilon = g$ on $\partial L^\varepsilon(t)$. Next, put

$$a^\varepsilon := \begin{cases} a & \text{in } B - L^\varepsilon(t) \\ 1 & \text{in } L^\varepsilon(t). \end{cases}$$

Then from the divergence form of the PDE (2.1) we expect that $\frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = a\frac{\partial g}{\partial \nu^\varepsilon}$ along $\partial L^\varepsilon(t)$, ν^ε denoting the outward pointing unit vector to $\partial L^\varepsilon(t)$. Hence

$$\frac{\partial w^\varepsilon}{\partial \nu^\varepsilon} = a\frac{\partial g}{\partial \nu^\varepsilon} \quad \text{on } \partial L^\varepsilon(t). \tag{3.15}$$

In summary, we expect w^ε to solve the PDE within $\partial L^\varepsilon(t)$, subject to the overdetermined boundary conditions (3.14), (3.15). This is a free boundary problem for w^ε and $L^\varepsilon(t)$.

3.4 Building $\mathbf{u}^\epsilon$, $\mathbf{L}^\epsilon$. We intend next finding for small $\varepsilon > 0$ an exact solution $u^\varepsilon, L^\varepsilon(t)$ of the free boundary problem (3.9), (3.14), (3.15) as a perturbation of the limiting solutions $u^0, L^0(t)$ for times $0 \le t < t_1$. Here t_1 denotes some time before the lake L^0 starts to overflow.

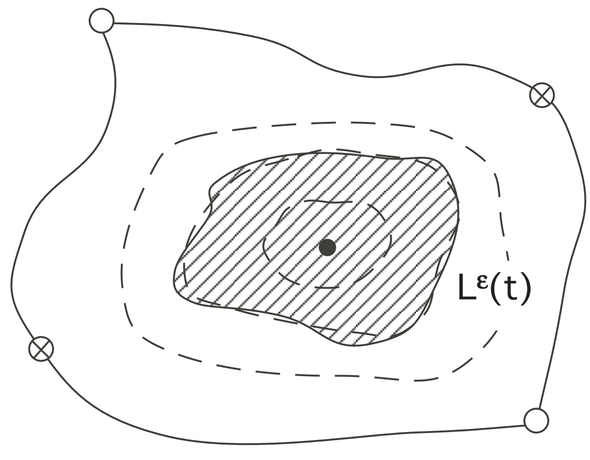

Figure 3: The lake $L^\epsilon(t)$ as a perturbation of the level curve $\{g = c(t)\}$

As compared with the much more subtle phenomena studied later in Sections 4 and 5, the problem at hand is essentially nonsingular, the only singularity occurring at the moment of the lakes formation, at the low point x_0. We do not in this paper attempt to analyze the advent of the lake at time 0. We prefer instead to bypass this issue, by supposing that we know the existence of a smooth solution $u^\varepsilon, L^\varepsilon$ for a short time interval $[0, t_0]$. We will extend the solution to the time interval $[t_0, t_1]$.

a. A functional equation. Set

$$\begin{aligned} L^0 &:= \{(x,t) \mid t_0 \le t \le t_1,\ x \in L^0(t)\}, \\ \Gamma^0 &:= \{(x,t) \mid t_0 \le t \le t_1,\ x \in \partial L^0(t)\}. \end{aligned}$$

Note that $|\nabla g|$ is bounded below near Γ^0. We propose to write

$$\partial L^\varepsilon(t) = \{x + \varepsilon\tau(x,t)\nu^0 \mid x \in \partial L^0(t)\} \tag{3.16}$$

for $t_0 \le t \le t_1$, where $\nu^0 := \frac{\nabla g}{|\nabla g|}$ is the outward pointing unit normal vector field to $\partial L^0(t) = \{g = c(t)\}$ and $\tau = \tau^\varepsilon$ must be found. We have $\tau = \tau_0^\varepsilon$ at time $t = t_0$, where τ_0^ε determines $\partial L^\varepsilon(t_0)$, which we assume is known. We also write

$$\begin{aligned} L^\varepsilon &:= \{(x,t) \mid t_0 \le t \le t_1,\ x \in L^\varepsilon(t)\}, \\ \Gamma^\varepsilon &:= \{(x,t) \mid t_0 \le t \le t_1,\ x \in \partial L^\varepsilon(t)\}. \end{aligned}$$

We introduce next for $0 < \alpha < 1$ the affine space

$$X := \{\tau : \Gamma^0 \to \mathbb{R} \mid \tau \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\Gamma^0), \tau = \tau_0^\varepsilon \text{ at } t = t_0\}.$$

To define the norm in X, we first write $\langle\,\cdot\,\rangle^{\varepsilon}_{\alpha} := [\,\cdot\,]_{C^{\alpha}_x} + \varepsilon^{\alpha/2}[\,\cdot\,]_{C^{\alpha/2}_t}$, where $[\,\cdot\,]_{C^{\alpha}_x}, [\,\cdot\,]_{C^{\alpha/2}_t}$ denote Hölder seminorms in the variables x and t. Now put

$$\|\tau\| := \max\{|\tau|_{L^\infty}, |\nabla^2_x\tau|_{L^\infty}, \varepsilon|\tau_t|_{L^\infty}, \langle\nabla_x\tau\rangle^{\varepsilon}_{\frac{1+\alpha}{2}}, \langle\nabla^2_x\tau\rangle^{\varepsilon}_{\alpha}, \varepsilon\langle\tau_t\rangle^{\varepsilon}_{\alpha}\}. \tag{3.17}$$

(Compare with the definition of parabolic Hölder spaces on page 46 of Lieberman [L].)

We will look for τ^ε as a solution of the following problem. Given $\tau \in X$, let w^ε solve

$$\begin{cases} \varepsilon w^\varepsilon_t - \Delta w^\varepsilon = f - \dot{c} & \text{in } L^\varepsilon \\ \frac{\partial w^\varepsilon}{\partial \nu^\varepsilon} = a\frac{\partial g}{\partial \nu^\varepsilon} & \text{on } \Gamma^\varepsilon, \\ w^\varepsilon = w^\varepsilon_0 & \text{on } L^\varepsilon \cap \{t = t_0\}, \end{cases} \tag{3.18}$$

where w^ε_0 is given at time $t = t_0$ and $\int_{L^\varepsilon(t_0)} w^\varepsilon_0\,dx = 0$. Next, define

$$T(\varepsilon,\tau) := w^\varepsilon - \left.\frac{g-c}{\varepsilon}\right|_{\partial L^\varepsilon}. \tag{3.19}$$

We want to find $\tau^\varepsilon \in X$ such that

$$T(\varepsilon,\tau^\varepsilon) \equiv 0. \tag{3.20}$$

By assumption this equality holds at $t = t_0$.

Define also

$$T(0,\tau) := w^0 - |\nabla g|\tau, \tag{3.21}$$

where w^0 solves

$$\begin{cases} -\Delta w^0 = f - \dot{c} & \text{in } L^0(t) \\ \frac{\partial w^0}{\partial \nu} = a|\nabla g| & \text{on } \partial L^0(t) \end{cases}$$

for each $t_0 \le t \le t_1$. In view of (3.12) there exists a unique solution satisfying the normalization $\int_{L^0(t)} w^0\,dx = 0$. Lastly, put

$$\tau^0 := \frac{w^0}{|\nabla g|} \quad \text{on } \Gamma^0;$$

so that $T(0,\tau^0) \equiv 0$.

b. Linearization. Proceeding formally, we write

$$0 = T(\varepsilon,\tau^\varepsilon) = T(\varepsilon,\tau^\varepsilon) - T(0,\tau^0) = T_\varepsilon(0,\tau^0)\varepsilon + T_\tau(0,\tau^0)(\tau^\varepsilon - \tau^0) + R, \tag{3.22}$$

for

$$R := T(\varepsilon,\tau^\varepsilon) - T(0,\tau^0) - T_\varepsilon(0,\tau^0)\varepsilon - T_\tau(0,\tau^0)(\tau^\varepsilon - \tau^0). \tag{3.23}$$

The definition (3.21) implies $T_\tau(0,\tau^0) = -|\nabla g|$; in which case (3.22) becomes

$$\tau^\varepsilon = \tau_0 + \frac{1}{|\nabla g|} T_\varepsilon(0,\tau^0)\varepsilon + \frac{R}{|\nabla g|} =: F(\tau^\varepsilon). \tag{3.24}$$

According to (3.19) we have

$$F(\tau) - F(\hat\tau) = \frac{T(\varepsilon,\tau) - T(\varepsilon,\hat\tau)}{|\nabla g|} + (\tau - \hat\tau),$$

where

$$\begin{cases} T(\varepsilon,\tau) = \left[w^\varepsilon - \frac{g-c}{\varepsilon}\right](x + \varepsilon\tau\nu^0) \\ T(\varepsilon,\hat\tau) = \left[\hat w^\varepsilon - \frac{g-c}{\varepsilon}\right](x + \varepsilon\hat\tau\nu^0), \end{cases}$$

w^ε solving (3.18) and $\hat w^\varepsilon$ solving

$$\begin{cases} \varepsilon \hat w^\varepsilon_t - \Delta \hat w^\varepsilon = f - \dot c & \text{in } \hat L^\varepsilon \\ \frac{\partial \hat w^\varepsilon}{\partial \nu^\varepsilon} = a\frac{\partial g}{\partial \nu^\varepsilon} & \text{on } \hat\Gamma^\varepsilon \\ \hat w^\varepsilon = w^\varepsilon_0 & \text{on } \hat L^\varepsilon \cap \{t = t_0\}. \end{cases} \tag{3.25}$$

Below we sketch a proof that $\|F(\tau) - F(\hat\tau)\| \le O(\varepsilon)\|\tau - \hat\tau\|$, provided

$$\|\tau\|, \|\hat\tau\| \le M \tag{3.26}$$

for some appropriate constant M. So for ε small enough, F will have a unique fixed point τ^ε.

c. Norm estimates. We record for later use some estimates involving the norm $\| \quad \|$, defined by (3.17).

Theorem 3.1 (i) *There exists a constant C, independent of ε, such that*

$$\|\tau\hat\tau\| \le C\|\tau\|\|\hat\tau\| \tag{3.27}$$

for all $\tau, \hat\tau \in X$.

(ii) *Suppose u solves the parabolic equation*

$$\begin{cases} \varepsilon u_t - \Delta u = f & \textit{in } L^0 \\ \frac{\partial u}{\partial \nu^0} = h & \textit{on } \Gamma^0 \\ u = u^0 & \textit{on } \partial L^0 \cap \{t = t_0\}. \end{cases} \tag{3.28}$$

Then

$$\|u\|_{2+\alpha} \leq C(\|f\|_\alpha + \|h\|_{1+\alpha} + \|u^0\|_{2+\alpha}), \tag{3.29}$$

for the scaled norms

$$\begin{aligned}
\|u\|_{2+\alpha} &:= \max\{|u|_{L^\infty}, |\nabla_x^2 u|_{L^\infty}, \varepsilon|u_t|_{L^\infty}, \langle \nabla_x u\rangle^\varepsilon_{\frac{1+\alpha}{2}}, \langle \nabla_x^2 u\rangle^\varepsilon_\alpha, \varepsilon\langle u_t\rangle^\varepsilon_\alpha\},\\
\|f\|_\alpha &:= \max\{|f|, \langle f\rangle^\varepsilon_\alpha\},\\
\|h\|_{1+\alpha} &:= \max\{|h|_{L^\infty}, |\nabla_x h|_{L^\infty}, \langle h\rangle^\varepsilon_{\frac{1+\alpha}{2}}, \langle \nabla_x h\rangle^\varepsilon_\alpha\},\\
\|u^0\|_{2+\alpha} &:= \max\{|u^0|_{L^\infty}, |\nabla_x^2 u^0|_{L^\infty}, [\nabla_x^2 u^0]_{C^\alpha_x}\}.
\end{aligned}$$

Proof. Assertion (i) follows from a direct calculation, the details of which we omit. To prove (ii), consider the function $\tilde{u}(x,t) = u(x,\varepsilon t)$ solves

$$\tilde{u}_t - \Delta\tilde{u} = \tilde{f}$$

for $\tilde{f}(x,t) = f(x,\varepsilon t)$, etc. We invoke the standard parabolic Schauder estimates, as in Theorem 4.3.1 of Lieberman [L], and rescale. □

d. Contraction estimates.

Theorem 3.2 *If* $\tau, \hat{\tau}$ *satisfy the bound* (3.26)*, then*

$$\|F(\tau) - F(\hat{\tau})\| \leq O(\varepsilon)\|\tau - \hat{\tau}\|. \tag{3.30}$$

Proof. 1. We begin by examining the terms involving g. We have

$$\begin{aligned}
\frac{g(x+\varepsilon\tau\nu^0) - g(x+\varepsilon\hat{\tau}\nu^0)}{\varepsilon} &= \int_0^1 \nabla g(x+\varepsilon(s\tau+(1-s)\hat{\tau})\nu^0)\cdot\nu^0 ds(\tau-\hat{\tau})\\
&= \int_0^1 [\nabla g(x+\varepsilon(s\tau+(1-s)\hat{\tau})\nu^0) - \nabla g(x)]\cdot\nu^0\, ds(\tau-\hat{\tau})\\
&\qquad + |\nabla g(x)|(\tau-\hat{\tau});
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{-g(x+\varepsilon\tau\nu^0) + g(x+\varepsilon\hat{\tau}\nu^0)}{\varepsilon} + |\nabla g(x)|(\tau-\hat{\tau})\\
&= -\varepsilon(\tau-\hat{\tau})\int_0^1\int_0^1 \nabla^2 g(x+r\varepsilon(s\tau+(1-s)\hat{\tau})\nu^0)(s\tau+(1-s)\hat{\tau})drds : (\nu^0\otimes\nu^0).
\end{aligned}$$

Using (3.27), we deduce

$$\left\| \frac{-g(x+\varepsilon\tau\nu^0)+g(x+\varepsilon\hat{\tau}\nu^0)}{\varepsilon} + |\nabla g|(\tau-\hat{\tau}) \right\| \leq O(\varepsilon)\|\tau-\hat{\tau}\|. \tag{3.31}$$

2. We must next estimate the term $w^\varepsilon(x+\varepsilon\tau\nu^0)-\hat{w}^\varepsilon(x+\varepsilon\hat{\tau}\nu^0)$. To do so, we change variables, shifting from L^ε and $\hat{L}^\varepsilon$ to the fixed region L^0. Given $\tau \in X$ we first extend τ to a small neighborhood N of Γ^0. Next, let $\boldsymbol{\nu}$ denote a smooth vector field that agrees with ν^0 on Γ^0 and vanishes outside N. Define $\boldsymbol{\phi}^\varepsilon : L^0 \to L^\varepsilon$ by

$$\boldsymbol{\phi}^\varepsilon(x,t) := x + \varepsilon\tau(x,t)\boldsymbol{\nu}(x,t),$$

and put $\boldsymbol{\psi}^\varepsilon := (\boldsymbol{\phi}^\varepsilon)^{-1}$. Define $v^\varepsilon = w^\varepsilon \circ \boldsymbol{\phi}^\varepsilon$. Then

$$\begin{cases} \varepsilon v_t^\varepsilon - \Delta v^\varepsilon = f^\varepsilon & \text{in } L^0 \\ \frac{\partial v^\varepsilon}{\partial \nu^0} = h^\varepsilon & \text{on } \Gamma^0 \\ v^\varepsilon = v_0^\varepsilon & \text{on } \partial L^0 \cap \{t=t_0\}, \end{cases}$$

where $v_0^\varepsilon = w_0^\varepsilon \circ \boldsymbol{\phi}^\varepsilon$,

$$f^\varepsilon := f(\boldsymbol{\phi}^\varepsilon) - v_{x_k}^\varepsilon(\varepsilon\psi_t^{\varepsilon,k} - \Delta\psi^{\varepsilon,k}) + (\psi_{x_i}^{\varepsilon,k}\psi_{x_i}^{\varepsilon,l} - \delta_{kl})v_{x_k,x_l}^\varepsilon - \dot{c},$$

and

$$h^\varepsilon := a^\varepsilon g_{x_k}^\varepsilon \nu_l \psi_{x_i}^{\varepsilon,k}\psi_{x_i}^{\varepsilon,l} + (\delta_{lk} - \psi_{x_i}^{\varepsilon,k}\psi_{x_i}^{\varepsilon,l})v_{x_k}^\varepsilon \nu_l$$

for $g^\varepsilon := g \circ \boldsymbol{\phi}^\varepsilon$, $a^\varepsilon := a \circ \boldsymbol{\phi}^\varepsilon$ and $\nu^0 = (\nu_1, \nu_2)$. The function $\hat{v}^\varepsilon := \hat{w}^\varepsilon \circ \hat{\boldsymbol{\phi}}^\varepsilon$ satisfies a similar equation.

Applying then (3.29), we have

$$\|v^\varepsilon - \hat{v}^\varepsilon\|_{2+\alpha} \leq C(\|f^\varepsilon - \hat{f}^\varepsilon\|_\alpha + \|h^\varepsilon - \hat{h}^\varepsilon\|_{1+\alpha}). \tag{3.32}$$

Now

$$\|f \circ \boldsymbol{\phi}^\varepsilon - f \circ \hat{\boldsymbol{\phi}}^\varepsilon\|_\alpha \leq C\varepsilon\|\tau - \hat{\tau}\|_\alpha \leq C\varepsilon\|\tau-\hat{\tau}\|.$$

Also

$$\begin{aligned} \|\nabla v^\varepsilon \cdot (\varepsilon\boldsymbol{\psi}_t^\varepsilon - \Delta\boldsymbol{\psi}^\varepsilon) - \nabla\hat{v}^\varepsilon \cdot (\varepsilon\hat{\boldsymbol{\psi}}_t^\varepsilon - \Delta\hat{\boldsymbol{\psi}}^\varepsilon)\|_\alpha &\leq C\varepsilon\|\nabla(v^\varepsilon - \hat{v}^\varepsilon)\|_\alpha \\ &\quad + C\|\varepsilon(\boldsymbol{\psi}^\varepsilon - \hat{\boldsymbol{\psi}}^\varepsilon)_t - \Delta(\boldsymbol{\psi}^\varepsilon - \hat{\boldsymbol{\psi}}^\varepsilon)\|_\alpha \\ &\leq C\varepsilon\|v^\varepsilon - \hat{v}^\varepsilon\|_{2+\alpha} + C\varepsilon\|\tau-\hat{\tau}\|, \end{aligned}$$

and

$$\begin{aligned} &\|\nabla^2 v^\varepsilon : (\nabla\boldsymbol{\psi}^\varepsilon \otimes \nabla\boldsymbol{\psi}^\varepsilon - I) - \nabla^2\hat{v}^\varepsilon : (\nabla\hat{\boldsymbol{\psi}}^\varepsilon \otimes \nabla\hat{\boldsymbol{\psi}}^\varepsilon - I)\|_\alpha \\ &\qquad \leq C\varepsilon\|\nabla^2(v^\varepsilon - \hat{v}^\varepsilon)\|_\alpha + C\|\nabla\boldsymbol{\psi}^\varepsilon \otimes \nabla\boldsymbol{\psi}^\varepsilon - \nabla\hat{\boldsymbol{\psi}}^\varepsilon \otimes \nabla\hat{\boldsymbol{\psi}}^\varepsilon\|_\alpha \\ &\qquad \leq C\varepsilon\|v^\varepsilon - \hat{v}^\varepsilon\|_{2+\alpha} + C\varepsilon\|\tau-\hat{\tau}\|. \end{aligned}$$

Consequently,

$$\|f^\varepsilon - \hat{f}^\varepsilon\|_\alpha \le C\varepsilon\|v^\varepsilon - \hat{v}^\varepsilon\|_{2+\alpha} + C\varepsilon\|\tau - \hat{\tau}\|. \tag{3.33}$$

Similarly, we have

$$\|h^\varepsilon - \hat{h}^\varepsilon\|_{1+\alpha} \le C\varepsilon\|v^\varepsilon - \hat{v}^\varepsilon\|_{2+\alpha} + C\varepsilon\|\tau - \hat{\tau}\|. \tag{3.34}$$

Combining estimates (3.32), (3.33), (3.34) and taking $\varepsilon > 0$ small enough, we deduce

$$\|v^\varepsilon - \hat{v}^\varepsilon\|_{2+\alpha} \le C\varepsilon\|\tau - \hat{\tau}\|.$$

It follows that

$$\|w^\varepsilon(\cdot + \varepsilon\tau\nu^0) - \hat{w}^\varepsilon(\cdot + \varepsilon\hat{\tau}\nu^0)\| \le C\varepsilon\|\tau - \hat{\tau}\|.$$

In view of this inequality and (3.31), the contraction estimate (3.30) is proved. □

Finally we must check that our construction yields a function u^ε satisfying $u^\varepsilon > g$ in L^ϵ.

Lemma 3.1 *Assume that*

$$0 \le a < \frac{1}{\varepsilon} \quad \text{on } \partial L^\epsilon(t),$$

Then for sufficiently small $\epsilon > 0$, we have

$$u^\epsilon > g \quad \text{within } L^\epsilon(t). \tag{3.35}$$

Proof. By construction we have $u^\varepsilon = c(t) + \varepsilon w^\varepsilon = g$ on $\partial L^\epsilon(t)$. Furthermore,

$$\frac{\partial(u^\varepsilon - g)}{\partial\nu^\varepsilon} = \varepsilon\frac{\partial w^\varepsilon}{\partial\nu^\varepsilon} - \frac{\partial g}{\partial\nu^\varepsilon} = (\varepsilon a - 1)\frac{\partial g}{\partial\nu^\varepsilon} < 0,$$

for small $\varepsilon > 0$, since $0 \le \varepsilon a < 1$ and $\lim_{\varepsilon\to 0}\frac{\partial g}{\partial\nu^\varepsilon} = \frac{\partial g}{\partial\nu} = |\nabla g| > 0$ on $\partial L^0(t)$. Hence $u^\varepsilon > g$ near $\partial L^\varepsilon(t)$; and the inequality (3.35) follows for small ε. □

4 Steady state problems

The following sections analyze how lakes overflow when they fill up. This turns out to be subtle, and we have been able to make progress only for the steady state case that f, u^0 and u^ε do not depend on the time variable t. In this section we introduce the steady state problem, make some elementary estimates, and discuss boundary condtions.

We hereafter assume that $u^\varepsilon = u^\varepsilon(x)$ satisfies

$$u^\varepsilon \ge g \tag{4.1}$$

and solves the stationary PDE

$$-\nabla \cdot (\frac{1}{\varepsilon}\nabla(u^\varepsilon - g) + a^\varepsilon \nabla g) = f \quad \text{in } \mathbb{R}^2 \tag{4.2}$$

for

$$a^\varepsilon \in \frac{\phi(u^\varepsilon - g)}{\varepsilon}. \tag{4.3}$$

4.1 Gradient estimates. First we show that u^ε is Lipschitz continuous, and in particular that the set

$$L^\varepsilon := \{u^\varepsilon > g\}$$

is open.

Lemma 4.1 *We have the uniform estimates*

$$||\nabla u^\varepsilon||_{L^\infty} \leq C \tag{4.4}$$

for some constant C*. In particular, we may assume*

$$u^\varepsilon \to u^0 \quad \textit{uniformly, as } \varepsilon \to 0.$$

Proof. 1. Since $v := u^\varepsilon - g$ solves the PDE

$$-\Delta v = \nabla \cdot (\varepsilon a^\varepsilon \nabla g) + \varepsilon f \tag{4.5}$$

and $0 \leq \varepsilon a^\varepsilon \leq 1$, we have the estimate $||v||_{W^{1,p}} \leq C$ for each $1 < p < \infty$. In particular v and so also u^ε are Hölder continuous for each exponent $0 < \alpha < 1$.

2. Next, let $x_0 \in L^\varepsilon$ and put $r := \operatorname{dist}(x_0, \partial L^\varepsilon)$. We may assume $0 \in \partial L^\varepsilon$ is a point on the boundary closest to x_0. Assuming for the moment v is smooth, we calculate

$$\frac{d}{ds} \fint_{B(0,s)} v \, dx = \frac{1}{2s} \fint_{B(0,s)} \Delta v (s^2 - |x|^2) \, dx,$$

the slash through the integral denoting an average. Substituting from (4.5) and integrating by parts, we deduce

$$\frac{d}{ds} \fint_{B(0,s)} v \, dx = -\frac{\varepsilon}{2s} \fint_{B(0,s)} f(s^2 - |x|^2) \, dx - \frac{1}{s} \fint_{B(0,s)} \varepsilon a^\varepsilon \nabla g \cdot x \, dx = O(1).$$

Since $v(0) = 0$, it follows that

$$\fint_{B(0,2r)} v \, dx \leq Cr.$$

We can mollify to make the same deduction if v is only a weak solution of (4.2).

Since $v \geq 0$, we have

$$\fint_{B(x_0,r)} v\,dx \leq Cr. \tag{4.6}$$

But $-\Delta v = \Delta g + \varepsilon f$ in $B(x_0, r) \subseteq L^\varepsilon$, and consequently (4.6) implies $0 \leq v(x_0) \leq Cr$. Thus v grows at most linearly away from ∂L^ε and solves the PDE $-\Delta v = \Delta g + \varepsilon f$ within L^ε: estimate (4.4) follows. □

4.2 Boundary conditions. We next uncover implicit in (4.2) boundary conditions for u^ε along ∂L^ε. Let a denote the function computed as in Section 3 from the surface flow equations (3.3), (3.4).

Lemma 4.2 *Suppose now that ∂L^ε is smooth and so $u^\varepsilon \in C^\infty(\overline{L^\varepsilon})$. Then*

$$u^\varepsilon = g, \quad \frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = \varepsilon a \left(\frac{\partial g}{\partial \nu^\varepsilon}\right)^+ - \left(\frac{\partial g}{\partial \nu^\varepsilon}\right)^- \quad \text{on } \partial L^\varepsilon. \tag{4.7}$$

As before, ν^ε denotes the outward unit normal to ∂L^ε and $x^+ = \max\{x, 0\}, x^- = -\min\{x, 0\}$.

Proof. Obviously $u^\varepsilon = g$ on ∂L^ε, and the divergence structure of the PDE (4.2) implies that

$$\frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = \varepsilon a^\varepsilon \frac{\partial g}{\partial \nu^\varepsilon}.$$

Consequently, to derive (4.7) we must show that $\varepsilon a^\varepsilon = 1$ if $\frac{\partial g}{\partial \nu^\varepsilon} < 0$. But if $\varepsilon a^\varepsilon < 1$, then $\frac{\partial (u^\varepsilon - g)}{\partial \nu^\varepsilon} = (\varepsilon a^\varepsilon - 1)\frac{\partial g}{\partial \nu^\varepsilon} > 0$. But this is impossible, since $u^\varepsilon > g$ in L^ε and $u^\varepsilon = g$ on ∂L^ε. □

For later reference, we introduce the notation

$$\Gamma^+ := \left\{x \in \partial L^\varepsilon \mid \frac{\partial g}{\partial \nu^\varepsilon} > 0\right\}, \quad \Gamma^- := \left\{x \in \partial L^\varepsilon \mid \frac{\partial g}{\partial \nu^\varepsilon} < 0\right\}. \tag{4.8}$$

Interpretation: L. Caffarelli has pointed out to us that the boundary conditions (4.7) present us with a nonstandard free boundary problem that is, in the case of no surface flow, a combination of an obstacle problem along Γ^+ and a prescribed normal derivative problem along Γ^-, as discussed for instance in Chapters 1 and 3 of Friedman [F].

We regard $\frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = \varepsilon a \left(\frac{\partial g}{\partial \nu^\varepsilon}\right)^+ \geq 0$ as the incoming boundary condition on Γ^+, for which any value of the flow rate $0 \leq a \leq \frac{1}{\varepsilon}$ is allowed. By contrast the outgoing boundary condition on

Γ^- is $\frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = -\left(\frac{\partial g}{\partial \nu^\varepsilon}\right)^- < 0$. This means that *water can exit the region L^ε only at the maximum flow rate $a^\varepsilon = \frac{1}{\varepsilon}$.* Furthermore,

$$\Gamma^- \subseteq \{\Delta g \leq 0\}; \tag{4.9}$$

as otherwise the ODE (3.3), (3.4) would immediately drive $a^\varepsilon > \frac{1}{\varepsilon}$, contradicting (4.3).

This picture of water flowing into the lake through Γ^+ and leaving through Γ^- is intuitively appealing, but we should be aware of the possibility that the water exits through one part of ∂L^ε, flows across the surface, and then reenters through another part of ∂L^ε. Furthermore we may not have a clean separation of the regions $\Gamma^\pm$, and so this exiting and reentering may possibly reoccur on very small length scales.

4.3 Convergence of $\mathbf{L}^\epsilon$. For this section we put

$$L^0 := \{u^0 > g\},$$

and let

$$R^0 := \{x \in \mathbb{R}^2 - L^0 \,|\nabla g(x) \neq 0, \text{the gradient flow from } x \text{ approaches a saddle point of } g \text{ as } \tau \to -\infty\}.$$

In other words, R^0 consists of the union of all gradient flow lines determined by the surface flow ODE (3.3) for which the uphill end of the curve approaches a saddle point (and not a local maximum) of g. Note that R^0 contains all the saddle points. As we will see, in the $\varepsilon \to 0$ limit such curves may support rivers originating as uphill lakes overflow through the saddle points.

Let us also define

$$\liminf_{\varepsilon\to 0} L^\varepsilon := \cup_{j=1}^\infty \cap_{\{0<\varepsilon<\frac{1}{j}\}} L^\varepsilon, \quad \limsup_{\varepsilon\to 0} L^\varepsilon := \cap_{j=1}^\infty \cup_{\{0<\varepsilon<\frac{1}{j}\}} L^\varepsilon.$$

Lemma 4.3 *We have*

$$|L^\varepsilon - L^0| \to 0 \tag{4.10}$$

as $\varepsilon \to 0$. Furthermore

$$L^0 \subseteq \liminf_{\varepsilon\to 0} L^\varepsilon. \tag{4.11}$$

We will see later that it is not true in general that $\limsup_{\varepsilon\to 0} L^\varepsilon \subseteq L^0$. We conjecture that

$$\limsup_{\varepsilon\to 0} L^\varepsilon \subseteq L^0 \cup R^0.$$

Proof. Let ζ denote a smooth cutoff function compactly supported in $\mathbb{R}^2 - L^0$. Then

$$\int_{L^\varepsilon} |\nabla(u^\varepsilon - g)|^2 \zeta^2 \, dx = -\int_{L^\varepsilon} (u^\varepsilon - g) \nabla \cdot (\zeta^2 \nabla(u^\varepsilon - g)) \, dx \to 0,$$

as $\varepsilon \to 0$, since $u^\varepsilon \to u^0 = g$ uniformly on $\mathbb{R}^2 - L^0$ and $-\Delta u^\varepsilon = \varepsilon f$ within L^ε.

Write

$$A^\varepsilon_\gamma := \{x \in L^\varepsilon, \text{dist } (x, L^0) \geq \gamma\} \cap \{|\nabla g| > \gamma\}$$

for $\gamma > 0$. Then

$$\gamma^2 |A^\varepsilon_\gamma| \leq \int_{A^\varepsilon_\gamma} |\nabla g|^2 \, dx \leq 2 \int_{L^\varepsilon} |\nabla u^\varepsilon|^2 \, dx + 2 \int_{A^\varepsilon_\gamma} |\nabla(u^\varepsilon - g)|^2 \, dx = o(1)$$

as $\varepsilon \to 0$. This proves (4.10).

The statement (4.11) is trivial, since $u^\varepsilon \to u^0$ uniformly. □

5 Lakes overflowing, rivers

In this section we study for a steady flow how a "lake overflows" at a saddle point of g, for both the cases $\varepsilon = 0$ and small $\varepsilon > 0$.

Geometry of $\mathbf{L^0}$. We assume that g has a nondegenerate saddle point at 0 and then as necessary change coordinates so that

(5.1) $$g(0) = 0, \; g_{x_1 x_2}(0) = 0, \; g_{x_1 x_1}(0) < 0, \; g_{x_2 x_2}(0) > 0.$$

We suppose the lake for $\varepsilon = 0$ lies in the right half plane:

$$L^0 = \{x \mid g(x) \leq 0, \; x_1 \geq 0\}.$$

According to our standing assumptions $\Delta g(0) \neq 0$ and ∂L^0 contains only the one critical point of g at 0. Hence ∂L^0 is smooth, except for a corner at the origin, of opening angle

(5.2) $$2\theta^0 = 2 \arctan \left(\frac{-g_{x_1 x_1}(0)}{g_{x_2 x_2}(0)} \right)^{1/2}.$$

See Figure 4. Clearly then $u^0 \equiv 0$ in L^0.

5.1 Overflow for $\boldsymbol{\epsilon}$=0. We consider the case that the lake L^ε is "overflowing", meaning that u^ε is of order ε in $L^\varepsilon \cap \{x_1 \geq 0\}$. More precisely, we suppose

(5.3) $$\int_{L^\varepsilon \cap \{x_1 \geq 0\}} (u^\varepsilon)^2 \, dx = O(\varepsilon^2).$$

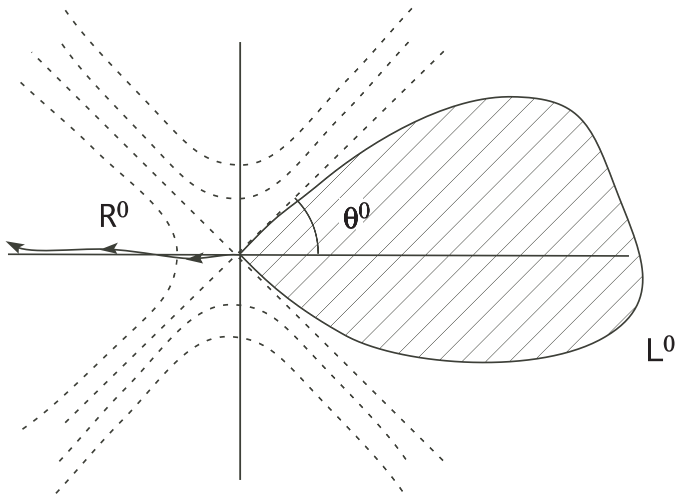

Figure 4: A lake overflowing, $\epsilon = 0$

We extract useful information by writing

$$u^\varepsilon = \varepsilon w^\varepsilon \quad \text{within } L^\varepsilon;$$

so that (4.2) implies

$$-\Delta w^\varepsilon = f \quad \text{within } L^\varepsilon. \tag{5.4}$$

We intend to show that $w^\epsilon \to w^0$, where

$$-\Delta w^0 = f \quad \text{within } L^0. \tag{5.5}$$

a. Boundary conditions for $\mathbf{w^0}$. By analogy with (4.7), we take the inflow boundary conditions

$$\frac{\partial w^0}{\partial \nu} = a\frac{\partial g}{\partial \nu} = a|\nabla g| \quad \text{on } \partial L^0 - \{0\}. \tag{5.6}$$

But this inflow must be balanced by an outflow at 0. Since solving (5.5) requires the compatibility condition $\int_{L^0} f \; dx = -\int_{\partial L^0} \frac{\partial w^0}{\partial \nu}\, ds$, we must have the singular boundary condition

$$\frac{\partial w^0}{\partial \nu}(0) = -a^*\delta_0, \tag{5.7}$$

where

$$a^* := \int_{L^0} f \; dx + \int_{\partial L^0 - \{0\}} a|\nabla g|\, ds \tag{5.8}$$

represents the total influx of water to our lake from, respectively, rainfall over the lake and surface flows into the lake. We interpret the boundary condition (5.7) to mean

$$\lim_{\delta\to 0}\int_{\partial B(0,\delta)\cap L^0}\frac{\partial w^0}{\partial\nu}\,ds=-a^*, \tag{5.9}$$

where ν is the inwardly pointing vector field along $\partial B(0,\delta)$.

b. Solving for $\mathbf{w^0}$. We next solve for w^0, paying particular attention to the singular boundary condition (5.7) at the corner of ∂L^0.

Theorem 5.1 (i) *There exists a unique solution of* (5.5), (5.6) *and* (5.7), *satisfying the normalizations*

$$\int_{L^0} w^0\,dx=0,\ \int_{L^0}(w^0)^2\,dx<\infty. \tag{5.10}$$

(ii) *Furthermore, w^0 has the asymptotic form*

$$w^0=A\log r+B+Cx_1+Dx_2+O(r^2)\qquad \text{as } r\to 0, \tag{5.11}$$

for $A:=\frac{a^*}{2\theta^0}$ and appropriate constants B,C,D, depending upon the second and third derivatives of g at 0. If g is even in the variable x_1, then $C=0$; if g is even in x_2, then $D=0$.

Proof. 1. Note first that in the sector $S:=\{|\theta|\le\theta^0, r>0\}$, the harmonic function $\log r$ satisfies the Neumann condition $\frac{\partial\log r}{\partial\nu}=\frac{\partial\log r}{\partial\theta}=0$ along the sides $\theta=\pm\theta^0$. Also

$$\int_{\partial B(0,\delta)\cap S}\frac{\partial\log r}{\partial\nu}\,ds=-2\theta^0.$$

2. We must examine the behavior of $\frac{\partial\log r}{\partial\nu}$ on ∂L^0. For this, let $x_2=\phi_\pm(x_1)$ denote the upper and lower branches of ∂L^0. Then $\phi'_\pm(0)=\pm\tan\theta^0$, and also

$$g_{x_1}+g_{x_2}\phi'_\pm\equiv 0,\quad g_{x_1x_1}+2g_{x_1x_2}\phi'_\pm+g_{x_2x_2}(\phi'_\pm)^2+g_{x_2}\phi''_\pm\equiv 0.$$

In light of (5.2) and (5.1), $g_{x_1x_1}+2g_{x_1x_2}\phi'_\pm+g_{x_2x_2}(\phi'_\pm)^2=0$ at $x_1=0$. Therefore we can use L'Hospital's rule to compute :

$$\phi''_\pm(0)=-\frac{g_{x_1x_1x_1}+3g_{x_1x_1x_2}\phi'_\pm(0)+3g_{x_1x_2x_2}\phi'_\pm(0)^2+g_{x_2x_2x_2}\phi'_\pm(0)^3}{3g_{x_2x_2}\phi'_\pm(0)}, \tag{5.12}$$

the partial derivatives of g evaluated at 0. Furthermore, along ∂L^0 we have

$$l_\pm = \lim_{x_1\to 0} \frac{\partial \log r}{\partial \nu} = \lim_{x_1\to 0} \frac{x\cdot\nu}{r^2} = \lim_{x_1\to 0} \frac{x\cdot\nabla g}{r^2|\nabla g|}$$
$$= \left(\lim_{x_1\to 0} \frac{g_{x_2}}{|\nabla g|}\right)\left(\lim_{x_1\to 0} \frac{-x_1\phi'_\pm + \phi_\pm}{r^2}\right) = \mp \frac{\cos\theta^0\,\phi''_\pm(0)}{2(1+\phi'_\pm(0)^2)}.$$

Using the calculation (5.12), we can therefore write

$$l_\pm = \alpha \pm \beta \tag{5.13}$$

for appropriate constants α, β, computable in terms of the second and third derivatives of g at $(0,0)$. (If g is even in the variable x_1, then $g_{x_1x_1x_1}, g_{x_1x_2x_2} = 0$ at the origin, and so (5.12) implies that $\alpha = 0$. Likewise, if g is even in x_2, then $g_{x_1x_1x_2}, g_{x_2x_2x_2} = 0$, and so $\beta = 0$.)

We select the constants $C = \frac{\alpha A}{\sin\theta^0}, D = -\frac{\beta A}{\cos\theta^0}$. Then

$$\lim_{x_1\to 0} \frac{\partial(A\log r + Cx_1 + Dx_2)}{\partial\nu} = 0, \tag{5.14}$$

along both branches of ∂L^0.

3. We will construct w^0 as $w^0 = A\log r + Cx_1 + Dx_2 + v^0$, where v^0 solves

$$\begin{cases} -\Delta v^0 = f & \text{in } L^0 \\ \frac{\partial v^0}{\partial \nu} = b & \text{on } \partial L^0 \end{cases} \tag{5.15}$$

for

$$b := a|\nabla g| - \frac{\partial(A\log r + Cx_1 + Dx_2)}{\partial\nu}. \tag{5.16}$$

According to (5.14), $\lim_{x_1\to 0} b = 0$ along both branches of ∂L^0.

Since $\int_{L^0} f\,dx = -\int_{\partial L^0} b\,ds$, the PDE (5.15) has a unique solution v^0 with zero mean, which we can build for instance by minimizing the functional

$$\int_{L^0} \frac{1}{2}|\nabla v|^2 - fv\,dx - \int_{\partial L^0} bv\,ds$$

among functions $v \in H^1(L^0)$ with zero mean. As $b \in C(\partial L^0)$, this functional is indeed defined on $H^1(L^0)$. Furthermore $\nabla v^0(0) = 0$, and so $v^0 = B + O(r^2)$ as $r \to 0$, for $B := v^0(0)$.

4. We finally show that our solution is unique. To do so, let w denote the difference of two solutions. Then

$$\begin{cases} \Delta w = 0 & \text{in } L^0 \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial L^0 - \{0\} \end{cases} \tag{5.17}$$

and

$$\int_{L^0} w\,dx = 0, \int_{L^0} w^2\,dx < \infty. \tag{5.18}$$

We want to deduce $w \equiv 0$, but this is not obvious since we do not know that $w \in H^1(L^0)$, only that $w \in H^1(L^0 - B(0,\delta))$ for each $\delta > 0$. Select any two distinct points $x_1, x_2 \in L^0$. Then let $v \in H^1_{loc}(L^0 - \{x_1, x_2\})$ solve

$$\begin{cases} \Delta v = \delta_{x_1} - \delta_{x_2} \quad \text{in } L^0 \\ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial L^0 - \{0\}. \end{cases} \tag{5.19}$$

The function v is smooth, and furthermore we have a sup-norm bound for ∇v, in any region bounded away from x_1, x_2.

Put $\Gamma_\delta := \partial B(0,\delta) \cap L^0$. Applying Green's identities in the region $L^0 - B(0,\delta)$ for small $\delta > 0$ and recalling (5.17), (5.19), we deduce that

$$w(x_1) - w(x_2) = \int_{\Gamma_\delta} \frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu}(v - v(0))\,ds.$$

We average the above identity, to find

$$w(x_1) - w(x_2) = -\frac{1}{\delta} \int_{\Delta_\delta} \nabla v \cdot \frac{x}{|x|} w - \nabla w \cdot \frac{x}{|x|}(v - v(0))\,dx =: A + B, \tag{5.20}$$

for $\Delta_\delta := L^0 \cap \{2\delta \le |x| \le 3\delta\}$.

Now

$$|A| \le \frac{1}{\delta} ||\nabla v||_{L^\infty} |\Delta_\delta|^{\frac{1}{2}} \left(\int_{\Delta_\delta} w^2\,dx \right)^{\frac{1}{2}} = o(1) \quad \text{as } \delta \to 0,$$

since $w \in L^2$. Furthermore,

$$|B| \le \frac{1}{\delta} ||v - v(0)||_{L^\infty(\Delta_\delta)} |\Delta_\delta|^{\frac{1}{2}} \left(\int_{\Delta_\delta} |\nabla w|^2\,dx \right)^{\frac{1}{2}} \le C\delta \left(\int_{\Delta_\delta} |\nabla w|^2\,dx \right)^{\frac{1}{2}}.$$

But since w solves (5.17), we have the estimate

$$\int_{\Delta_\delta} |\nabla w|^2\,dx \le \frac{C}{\delta^2} \int_{\tilde{\Delta}_\delta} w^2\,dx$$

for $\tilde{\Delta}_\delta := L^0 \cap \{\delta \le |x| \le 4\delta\}$. We combine the previous two inequalities, to deduce

$$|B| \le C \int_{\tilde{\Delta}_\delta} w^2\,dx = o(1) \quad \text{as } \delta \to 0.$$

Letting $\delta \to 0$ in (5.20), we conclude that $w(x_1) = w(x_2)$ for all points x_1, x_2. Hence w is identically constant; and in light of (5.18), in fact $w \equiv 0$. □

Remark. For each integer k, the functions

$$v := r^{a_k} \cos(a_k\theta), r^{b_k} \sin(b_k\theta) \quad \text{for } a_k := k\frac{\pi}{\theta^0},\, b_k := \left(k + \frac{1}{2}\right)\frac{\pi}{\theta^0}$$

are harmonic in the sector S and satisfy the Neumann condition $\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \theta} = 0$ along the sides $\theta = \pm\theta^0$. However for $k < 0$ these functions do not belong to $L^2_{loc}(S)$. Thus the singularity of w^0 is logarithmic, and not worse. □

5.2 Convergence as $\boldsymbol{\epsilon \to 0}$. Next we investigate the limiting behavior of w^ε and ∂L^ε as $\varepsilon \to 0$.

a. Convergence of $\mathbf{w}^\epsilon$ to $\mathbf{w}^0$. Define $c^\varepsilon := \int_{L^\varepsilon} w^\varepsilon \, dx$. According to (5.3), we have the estimate

$$\int_{L^\varepsilon \cap \{x_1 \geq 0\}} (w^\varepsilon)^2 \, dx \leq C; \tag{5.21}$$

and so in particular the constants $\{c^\varepsilon\}_{0 \leq \varepsilon \leq 1}$ are bounded.

We go further we need to make the *geometric assumption* that for each $\delta > 0$:

$$|(L^0 \Delta L^\varepsilon) \cap \{x_1 \geq \delta\}| \leq C_\delta \varepsilon, \tag{5.22}$$

where Δ here denotes the symmetric difference and the constant C_δ is independent of ε. This says that away from possible singular effects at the origin, the lakes L^ε and L^0 differ in area by $O(\varepsilon)$. This bound is consistent with the construction of ∂L^ε as a perturbation of ∂L^0 of order ε presented in Section 3. Furthermore we can prove (5.22) if we make some reasonable assumptions about the geometry of $\partial L^\varepsilon \cap \{x_1 \geq \delta\}$:

Theorem 5.2 *Assume that within each region $\{x_1 \geq \delta\}$ the free boundary ∂L^ε is a $C^{1,1}$ graph over ∂L^0, the bounds controlling the $C^{1,1}$-norms independent of ε.*

Then estimate (5.22) *holds.*

Proof. Define the region $L^\varepsilon_\delta := L^\varepsilon \cap \{x_1 > \frac{\delta}{2}\}$, and let v^ε solve

$$\begin{cases} -\Delta v^\varepsilon = 1 & \text{within } L^\varepsilon_\delta \\ v^\varepsilon = 0 & \text{on } \partial L^\varepsilon_\delta. \end{cases} \tag{5.23}$$

Our hypotheses imply

$$\frac{\partial v^\varepsilon}{\partial \nu} \leq -a < 0 \text{ on } \partial L^\varepsilon \cap \{x_1 \geq \delta\} \tag{5.24}$$

for some positive constant a that does not depend upon ε. Green's identity gives

$$\int_{\partial L^\varepsilon_\delta} (w^\varepsilon)^2 \frac{\partial v^\varepsilon}{\partial \nu} ds = \int_{L^\varepsilon_\delta} (w^\varepsilon)^2 \Delta v^\varepsilon - v^\varepsilon \Delta (w^\varepsilon)^2 \, dx,$$

the right hand side of which is bounded. Owing now to (5.24), we deduce that

$$\frac{1}{\varepsilon^2} \int_{\partial L^\varepsilon \cap \{x_1 \geq \delta\}} g^2 \, ds = \int_{\partial L^\varepsilon \cap \{x_1 \geq \delta\}} (w^\varepsilon)^2 \, ds \leq C.$$

But $|g| \geq \gamma \mathrm{dist}(\partial L^\varepsilon, \partial L^0)$ in the region $\{x_1 \geq \delta\}$ for some positive constant γ; and consequently

$$\int_{\partial L^\varepsilon \cap \{x_1 \geq \delta\}} \mathrm{dist}(\partial L^\varepsilon, \partial L^0) \, ds \leq C\varepsilon.$$

This implies (5.22). □

Next, we demonstrate the convergence of w^ε to w^0:

Theorem 5.3 *Assume* (5.22). *Then*

(5.25) $$w^\epsilon - c^\varepsilon \to w^0 \quad \textit{in } L^2_{loc}, \textit{as } \varepsilon \to 0.$$

Proof. 1. Multiply (4.2) by $u^\varepsilon \zeta^2$, where ζ is a smooth cutoff function with compact support that vanishes for $x_1 \leq \delta$. We integrate by parts to find

(5.26) $$\int_{L^\varepsilon} \nabla u^\varepsilon \cdot \nabla (u^\varepsilon \zeta^2) \, dx + \int_{\mathbb{R}^2 - L^\varepsilon} \varepsilon a \nabla g \cdot \nabla (g\zeta^2) \, dx = \int_{\mathbb{R}^2} \varepsilon f u^\varepsilon \zeta^2 \, dx.$$

Now (5.3) implies

(5.27) $$\int_{L^\varepsilon} \varepsilon f u^\varepsilon \zeta^2 \, dx = O(\varepsilon^2).$$

We can furthermore use (5.22) and (3.1) to compute

$$\begin{aligned}
\int_{\mathbb{R}^2 - L^\varepsilon} \varepsilon f u^\varepsilon \zeta^2 \, dx &= \int_{\mathbb{R}^2 - L^0} \varepsilon f g \zeta^2 \, dx + O(\varepsilon^2) \\
&= -\int_{\mathbb{R}^2 - L^0} \varepsilon \nabla \cdot (a \nabla g) g \zeta^2 \, dx + O(\varepsilon^2) \\
&= \int_{\mathbb{R}^2 - L^0} \varepsilon a \nabla g \cdot \nabla (g\zeta^2) \, dx + O(\varepsilon^2) \\
&= \int_{\mathbb{R}^2 - L^\varepsilon} \varepsilon a \nabla g \cdot \nabla (g\zeta^2) \, dx + O(\varepsilon^2).
\end{aligned}$$

Note that $g = 0$ on ∂L^0 and so there is no boundary term when we integrate by parts. Employing this computation and (5.27) in (5.26), we deduce

$$\int_{L^\varepsilon} \nabla u^\varepsilon \cdot \nabla(u^\varepsilon \zeta^2)\, dx = O(\varepsilon^2);$$

whence routine calculations imply

$$\int_{L^\varepsilon} |\nabla w^\varepsilon|^2 \zeta^2\, dx \leq C_\delta. \tag{5.28}$$

2. Using the estimates we may assume

$$w^\varepsilon - c^\varepsilon \rightharpoonup w^0 \ \text{ weakly in } L^2, \quad w^\varepsilon - c^\varepsilon \rightharpoonup w^0 \ \text{ weakly in } H^1_{loc},$$

convergence in "H^1_{loc}" meaning in regions bounded away from 0.

We next claim that the limit function w^0 solves the singular boundary value problem (5.5), (5.6) and (5.7); and therefore w^0 is the function constructed in Theorem 5.1. Let v denote a smooth function with compact support, vanishing for $x_1 \leq \delta$. Multiply (4.2) by v and integrate by parts:

$$\int_{L^\varepsilon} \nabla w^\varepsilon \cdot \nabla v\, dx + \int_{\mathbb{R}^2 - L^\varepsilon} a \nabla g \cdot \nabla v\, dx = \int_{\mathbb{R}^2} f v\, dx.$$

As in the foregoing calculations, we recall (5.22) and (3.1), integrate by parts in the region $\mathbb{R}^2 - L^0$, and compute

$$\int_{L^\varepsilon} \nabla w^\varepsilon \cdot \nabla v\, dx - \int_{\partial L^0} a v \nabla g \cdot \nu^0\, ds = \int_{L^0} f v\, dx + O(\varepsilon^2),$$

for $\nu^0 = \nabla g / |\nabla g|$. Pass to limits as $\varepsilon \to 0$, to discover that

$$\int_{L^0} \nabla w^0 \cdot \nabla v\, dx - \int_{\partial L^0} a |\nabla g| v\, ds = \int_{L^0} f v\, dx$$

for all test functions v as above that vanish near 0. This identity means that w^0 is a weak, and therefore smooth, solution of $-\Delta w^0 = f$ in L^0, with $\frac{\partial w^0}{\partial \nu} = a|\nabla g|$ on $\partial L^0 - \{0\}$. The singular boundary condition (5.7), interpreted in the sense of (5.9), follows from the Divergence Theorem on $L^0 - B(0, \delta)$. □

b. The boundary of $\mathbf{L}^\epsilon$ away from 0. Next we show that the function w^0 controls the asymptotics of ∂L^ε as a perturbation of ∂L^0, at least away from the singularity at the origin.

Theorem 5.4 *Suppose $x^\varepsilon \in \partial L^\varepsilon$ can be written as $x^\varepsilon = x + \varepsilon\tau^\varepsilon\nu^0$ for some $x \in \partial L^0 - \{0\}$. Assume $w^\varepsilon \to w^0$ uniformly away from the origin. Then*

$$\tau^\varepsilon \to \tau^0 := \frac{w^0(x)}{|\nabla g(x)|}, \tag{5.29}$$

for the function w^0 introduced in Theorem 5.1.

Proof. We have

$$u^\varepsilon(x + \varepsilon\tau^\varepsilon\nu^0) = g(x + \varepsilon\tau^\varepsilon\nu^0) = g(x + \varepsilon\tau^\varepsilon\nu^0) - g(x) = |\nabla g(x)|\varepsilon\tau^\varepsilon + o(\varepsilon),$$

since $g = 0, \nu^0 = \nabla g/|\nabla g|$ on ∂L^0. As $u^\varepsilon = \varepsilon w^\varepsilon$, (5.29) holds. □

c. Convergence of surface flows, a river for ϵ=0. We construct next a one-dimensional river R^0 that flows downhill and to the left from the saddle point at 0. See Figure 4. For this section we redefine R^0, now to denote the path of the downhill gradient flow to the left away from the saddle point. We assume also that $f \equiv 0$ for $x_1 \le 0$; so that all of the rainfall occurs to the right of the origin.

We want to show that certain quantities controlling the flow rates converge, and will need to make a mild geometric assumption. For this, first select a small $\delta > 0$ and then find constants $l > 0, \gamma > 0$ so that $|\nabla g| \ge \gamma$ for $-l \le x_1 \le -\delta$ and for small $|x_2|$. We will suppose that the set $\{u^\varepsilon > g\}$ to the left of the origin has *approximately constant width* within the region $\{-l \le x_1 \le -\delta\}$. This means that for some small, fixed constant $\sigma > 0$, we have the estimate

$$|R^\varepsilon| \le C|R^\varepsilon_\sigma|, \tag{5.30}$$

where

$$R^\varepsilon := \{u^\varepsilon > g, -l \le x_1 \le -\delta, |x_2| \text{ small}\},$$
$$R^\varepsilon_\sigma := \{u^\varepsilon > g, -l + \sigma \le x_1 \le -\delta - \sigma, |x_2| \text{ small}\},$$

The constant in (5.30) depends on δ and σ, but not ε.

Lemma 5.1 *Under the geometric assumption* (5.30)*, we have the uniform estimate*

$$|R^\varepsilon| = O(\varepsilon). \tag{5.31}$$

Proof. Take a cutoff function $\zeta = \zeta(x_1)$ vanishing near $x_1 = -l, -\delta$, multiply the PDE (4.2) by $(u^\varepsilon - g)\zeta^2$, integrate by parts over R^ε, and make some simple estimates, to discover

$$\int_{R^\varepsilon} |\nabla(u^\varepsilon - g)|^2\zeta^2\,dx \le C\int_{R^\varepsilon} |u^\varepsilon - g|\zeta^2 + |\nabla\zeta|^2(u^\varepsilon - g)^2\,dx.$$

Take $\zeta \equiv 1$ for $-l+\sigma \leq x_1 \leq -\delta-\sigma$; then

$$\int_{R^\varepsilon_\sigma} |\nabla(u^\varepsilon - g)|^2\,dx \leq C|R^\varepsilon| \max_{R^\varepsilon}|u^\varepsilon - g| \leq C|R^\varepsilon_\sigma| \max_{R^\varepsilon}|u^\varepsilon - g|,$$

according to (5.30). Recalling that $|\nabla g| \geq \gamma > 0$ on R^ε, we deduce that

$$|R^\varepsilon_\sigma| \leq C\int_{R^\varepsilon_\sigma} |\nabla g|^2\,dx \leq C\int_{R^\varepsilon_\sigma} |\nabla(u^\varepsilon - g)|^2 + |\nabla u^\varepsilon|^2\,dx \leq C|R^\varepsilon_\sigma| \max_{R^\varepsilon}|u^\varepsilon - g| + C\varepsilon.$$

Since $\max|u^\varepsilon - g| \to 0$ as $\varepsilon \to 0$, we conclude for small ε that $|R^\varepsilon_\sigma| \leq C\varepsilon$ and hence that $|R^\varepsilon| \leq C\varepsilon$. □

Theorem 5.5 *In the region* $\{-l < x_1 < -\delta\}$, *we have*

$$\frac{1}{\varepsilon}\nabla u^\varepsilon \chi_{\{u^\varepsilon > g\}} + a^\epsilon \nabla g \chi_{\{u^\varepsilon = g\}} \rightharpoonup a^0 \nabla g \quad \textit{weakly as measures,} \tag{5.32}$$

as $\varepsilon \to 0$, *for*

$$a^0 := \frac{a^*}{|\nabla g|}\mathcal{H}^1 \llcorner R^0, \tag{5.33}$$

the constant a^* *defined by* (5.8).

Here $\mathcal{H}^1$ denotes one-dimensional Hausdorff measure and the symbol $\llcorner$ means restriction. We can therefore reinterpret a^* as the total flow along R^0: the total influx of water into the lake equals the total outflow in the river leaving the lake.

Proof. 1. First, multiply (4.2) by u^ε and integrate by parts, to estimate

$$\int_{\{u^\varepsilon > g\}} \frac{1}{\varepsilon}|\nabla u^\varepsilon|^2\,dx + \int_{\{u^\varepsilon = g\}} a^\varepsilon |\nabla g|^2\,dx \leq C. \tag{5.34}$$

2. In view of inequalities (5.34) and (5.31) the vector functions

$$\boldsymbol{\xi}^\varepsilon := \frac{1}{\varepsilon}\nabla u^\varepsilon \chi_{\{u^\varepsilon > g\}} + a^\epsilon \nabla g \chi_{\{u^\varepsilon = g\}}$$

are bounded in L^1 in the region $\{-l < x_1 < -\delta\}$. Consequently, passing as necessary to a subsequence, we may assume that $\boldsymbol{\xi}^\varepsilon \rightharpoonup \boldsymbol{\xi}$ weakly as measures, for some vector measure $\boldsymbol{\xi}$ supported in R^0. Our task is to identify $\boldsymbol{\xi}$.

Using the PDE (4.2), we see that

$$\int_{R^0} \nabla\phi \cdot d\boldsymbol{\xi} = \int_{\mathbb{R}^2} f\phi\,dx. \tag{5.35}$$

for each smooth function ϕ with compact support in $\{-l < x_1 < -\delta\}$. Also, (4.2) implies

$$\int_{\mathbb{R}^2} (\frac{1}{\varepsilon}\nabla(u^\varepsilon - g) + a^\varepsilon \nabla g) \cdot \nabla\phi \, dx = \int_{\mathbb{R}^2} \boldsymbol{\xi}^\varepsilon \cdot \nabla\phi \, dx = \int_{\mathbb{R}^2} f\phi \, dx.$$

But

$$\int_{\{u^\varepsilon > g\}} \frac{1}{\varepsilon}\nabla(u^\varepsilon - g) \cdot \nabla\phi \, dx = -\int_{\{u^\varepsilon > g\}} \frac{1}{\varepsilon}(u^\varepsilon - g) \cdot \Delta\phi \, dx = o(1), \tag{5.36}$$

since $u^\varepsilon \to g$ uniformly on the support of ϕ and estimate (5.31) holds. Consequently, we see that in fact

$$a^\varepsilon \nabla g \rightharpoonup \boldsymbol{\xi}, \tag{5.37}$$

where we recall that $a^\varepsilon = \frac{1}{\varepsilon}$ on $\{u^\varepsilon > g\}$. Since the functions a^ε are bounded in L^1 in the region $\{-l < x_1 < -\delta\}$, we deduce that

$$\boldsymbol{\xi} = a^0 \nabla g,$$

for some nonnegative measure a^0 with support in R^0. According to (5.35), we have

$$\int_{R^0} \nabla\phi \cdot \nabla g \, da^0 = \int_{\mathbb{R}^2} f\phi \, dx \tag{5.38}$$

for all functions ϕ as before.

3. Let ζ denote a smooth cutoff function vanishing outside of $\{-l < x_1 < -\delta\}$. Take $a < b < 0$. Define for small $\sigma_1, \sigma_2 > 0$ the function

$$\Phi(z) = \begin{cases} 0 & z \le a \\ 1 & a + \sigma_1 \le z \le b - \sigma_2 \\ 0 & z \ge b, \end{cases}$$

and take Φ to be linear and continuous on $[a, a+\sigma_1]$ and $[b - \sigma_2, b]$. We put $\phi = \Phi(g)\zeta$ in (5.38), to deduce

$$\begin{aligned} \frac{1}{\sigma_1} \int_{R^0 \cap \{a \le g \le a+\sigma_1\}} |\nabla g|^2 \zeta \, da^0 = \frac{1}{\sigma_2} &\int_{R^0 \cap \{b-\sigma_2 \le g \le b\}} |\nabla g|^2 \zeta \, da^0 \\ &- \int_{R^0} \nabla g \cdot \nabla\zeta \Phi(g) \, d\mu + \int_{\mathbb{R}^2} f\Phi(g)\zeta \, dx. \end{aligned}$$

Select a sequence of smooth functions $\zeta = \zeta_k$ as above, such that $\zeta_k \to 1$ on $R^0 \cap \{-l < x_1 < -\delta\}$, $\zeta_k \to 0$ on $\mathbb{R}^2 - R^0$. We discover then that

$$\frac{1}{\sigma_1} \int_{R^0 \cap \{a \le g \le a+\sigma_1\}} |\nabla g|^2 \, da^0 = \frac{1}{\sigma_2} \int_{R^0 \cap \{b-\sigma_2 \le g \le b\}} |\nabla g|^2 \, da^0. \tag{5.39}$$

Our sending $\sigma_1 \to 0$, while σ_2 is held fixed, shows that a^0 is absolutely continuous with respect to $\mathcal{H}^1$:

$$a^0 = \alpha \mathcal{H}^1 \llcorner R^0$$

for a function α locally summable with respect to $\mathcal{H}^1 \llcorner R^0$. Now let $\sigma_1, \sigma_2 \to 0$ in (5.39). We conclude that for almost all values of a, b that

$$\alpha|\nabla g|(R^0 \cap \{g = a\}) = \alpha|\nabla g|(R^0 \cap \{g = b\}).$$

It follows that $\alpha|\nabla g| \equiv \hat{a}$ is constant along R^0.

4. We will next identify the constant $\hat{a}$. To do so, select a smooth function ϕ such that $\phi \equiv 1$ for $x_1 \geq -\delta$ and $\phi \equiv 0$ for $x_1 \leq -l$. Select a small number $\lambda > 0$, multiply (4.2) by ϕ and integrate over the region $\{x_1 < \lambda\}$:

$$\begin{aligned}
\int_{\{x_1 < -\delta\}} \boldsymbol{\xi}^\varepsilon \cdot \nabla \phi \, dx &= \int_{\{x_1 < \lambda\}} f\phi \, dx + \int_{\partial L^\varepsilon \cap \{x_1 = \lambda\}} \boldsymbol{\xi}_1^\varepsilon \, ds \\
&= \int_{\{x_1 < \lambda\}} f\phi \, dx + \int_{\partial L^\varepsilon \cap \{x_1 \geq \lambda\}} f \, dx + \frac{1}{\varepsilon} \int_{\partial L^\varepsilon \cap \{x_1 \geq \lambda\}} \frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} \, ds \\
&\leq \int_{\{x_1 < \lambda\}} f\phi \, dx + \int_{\partial L^\varepsilon \cap \{x_1 \geq \lambda\}} f \, dx + \int_{\partial L^\varepsilon \cap \{x_1 \geq \lambda\}} \frac{\partial g}{\partial \nu^\varepsilon} \, ds \\
&= \int_{\{x_1 < \lambda\}} f\phi \, dx + a^* + o(1).
\end{aligned}$$

Letting $\varepsilon \to 0$ and then $\lambda \to 0$, we deduce

$$\int_{R^0} \alpha \nabla g \cdot \nabla \phi \, d\mathcal{H}^1 = a^*. \tag{5.40}$$

For $a < 0$ and for small $\sigma > 0$ define the function

$$\Phi(z) = \begin{cases} 0 & z \leq a \\ 1 & z \geq a + \sigma, \end{cases}$$

and extend Φ to be linear and continuous on $[a, a+\sigma]$. Let $\phi = \Phi(g)$ in the region $\{a \leq g \leq a + \sigma, x_1 < 0\}$, and extend ϕ to be zero on the left, one on the right. Then (5.40) implies

$$\frac{1}{\sigma} \int_{R^0 \cap \{a \leq g \leq a+\sigma\}} \alpha |\nabla g|^2 \, d\mathcal{H}^1 = a^*.$$

Since $\alpha|\nabla g| \equiv \hat{a}$, we deduce upon sending $\sigma \to 0$ that $\hat{a} = a^*$. □

5.3 Overflow for $\boldsymbol{\epsilon}$>0. We next attempt to understand the overflow from the lake L^ε as creating rivers and/or surface flows that are perturbations of the flow a^0 along the river

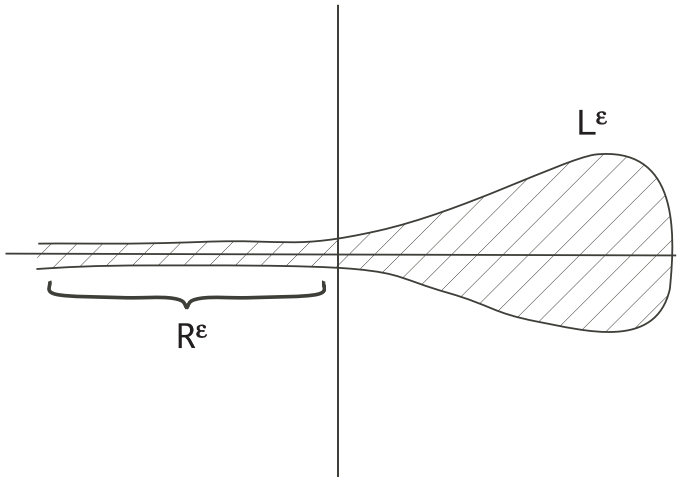

Figure 5: An overflowing lake and river, for $\epsilon > 0, \Delta g(0) > 0$.

R^0. This is a highly singular problem, and the behavior of L^ε, etc., near the saddle point 0 depends strongly upon the sign of $\Delta g(0) \neq 0$.

If $\Delta g(0) > 0$, then $g_{x_2x_2}(0) > -g_{x_1x_1}(0) > 0$ and so the overflow is from a shallow lake, through a mountain pass along the x_2-axis, with steep sides. If instead $\Delta g(0) < 0$, then $-g_{x_1x_1}(0) > g_{x_2x_2}(0) > 0$ and the overflow is from a deep lake, through a mountain pass with sides that are not steep. We discuss these cases in order.

Assume first that $\Delta g(0) > 0$ and that $f \equiv 0$ near 0. We argue heuristically that the lake $L^\varepsilon = \{u^\varepsilon > g\}$ has the form shown in Figure 5. Define the *approximate river*

$$R^\varepsilon := \{u^\varepsilon > g\} \cap \{x_1 < 0\}.$$

First note that since $\Delta g(0) > 0$, we have $0 < \theta^0 < \frac{\pi}{4}$ for the opening of the lake L^0 at 0. We observe next that if ∂L^ε is smooth, then

$$\frac{\partial g}{\partial \nu^\varepsilon} > 0, \ 0 \leq \varepsilon a < 1 \quad \text{on } \partial L^\varepsilon. \tag{5.41}$$

To see this, we observe that

$$\begin{cases} -\Delta(u^\varepsilon - g) = \Delta g > 0 & \text{in } L^\varepsilon \\ \qquad\quad u^\varepsilon - g > 0 & \text{in } L^\varepsilon \\ \qquad\quad u^\varepsilon - g = 0 & \text{on } \partial L^\varepsilon. \end{cases}$$

Hence Hopf's Lemma implies $\frac{\partial(u^\varepsilon - g)}{\partial \nu^\varepsilon} < 0 \quad$ on ∂L^ε. Thus $\frac{\partial g}{\partial \nu^\varepsilon} > \frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = \frac{\varepsilon \partial w^\varepsilon}{\partial \nu^\varepsilon} = a\frac{\partial g}{\partial \nu^\varepsilon}$, the last equality holding according to (3.15). Since $0 \leq a \leq 1$, the inequalities (5.41) hold.

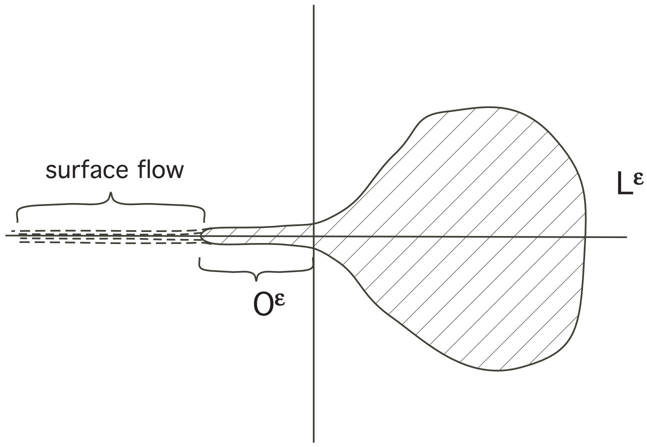

Figure 6: An overflowing lake for $\epsilon > 0, \Delta g(0) < 0$.

According to the first inequality in (5.41) *the boundary* ∂L^ε *is transverse to the flow lines of the ODE* (3.3)*. In particular, the approximate river* R^ε *cannot terminate within the region where* $\Delta g > 0$.

We argue heuristically that the asymptotic width of the river R^ε should be

$$d^\varepsilon = \frac{\varepsilon a^*}{|\nabla g|}(1 + o(1)) \quad \text{as } \varepsilon \to 0. \tag{5.42}$$

To see this, recall from (5.37) that $a^\varepsilon \nabla g \rightharpoonup \boldsymbol{\xi} = a^* \frac{\nabla g}{|\nabla g|} \mathcal{H}^1 \llcorner R^0$. Thus

$$\frac{1}{\varepsilon} \int_{R^\varepsilon} \nabla g \cdot \nabla \phi \, dx \to \int_{R^0} a^* \nabla \phi \cdot \frac{\nabla g}{|\nabla g|} \, d\mathcal{H}^1. \tag{5.43}$$

Using the Coarea Formula, we write

$$\begin{aligned} \frac{1}{\varepsilon} \int_{R^\varepsilon} \nabla g \cdot \nabla \phi \, dx &= \int \left(\frac{1}{\varepsilon} \int_{\{g=s\} \cap R^\varepsilon} \nabla \phi \cdot \frac{\nabla g}{|\nabla g|} \, d\mathcal{H}^1 \right) ds \\ &= \int \frac{d^\varepsilon}{\varepsilon} \nabla \phi \cdot \frac{\nabla g}{|\nabla g|} \, ds + o(1) \\ &= \int_{R^0} \frac{d^\varepsilon}{\varepsilon} \nabla \phi \cdot \nabla g \, d\mathcal{H}^1 + o(1). \end{aligned}$$

This formula, valid for all ϕ as above, implies in light of (5.43) the asymptotic estimate (5.42).

Now suppose that $\Delta g(0) < 0$ and that $f \equiv 0$ near 0. We conjecture in this case that the outflow from the lake L^ε cannot be a thin river (where $u^\varepsilon > g$), but rather consists of a short

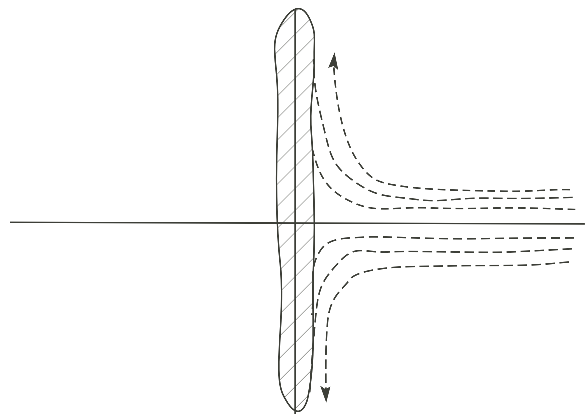

Figure 7: Surface flow and “temporary lake”, $\Delta g > 0$.

outlet O_ε, followed by a surface flow (where $u^\varepsilon = g$), as shown in Figure 6. We however have not been able to come up with any either rigorous or heuristically convincing asymptotics for the behavior of ∂L^ε near the orgin as $\varepsilon \to 0$. Figure 9 at the end of this paper displays some numerical simulations that seem consistent with our conjectures.

5.4 Rivers and surface flows into saddle points. We have seen that our scenario of lakes filling up and overflowing is fairly simple in the $\varepsilon \to 0$ limit and less so for $\varepsilon > 0$. One phenomenon is however puzzling even for $\varepsilon = 0$: what happens when a one-dimensional river flows downhill not into a lake, but rather a saddle point? In this section we provide some conjectures about what happens, depending upon the sign of Δg at the saddle.

Recall from (3.5) that

$$a^\varepsilon(\boldsymbol{\gamma}(\tau)) = a^\varepsilon(\boldsymbol{\gamma}(\tau_0))e^{\int_{\tau_0}^{\tau} \Delta g(\boldsymbol{\gamma}(s))ds}$$

in any region where $f \equiv 0$. Suppose the downhill flow approaches a saddle point at 0, where $\Delta g(0) > 0$. Those trajectories close to the x_2-axis will spend a long time near 0 and consequently the term $e^{\int_{t_0}^{t} \Delta g(\gamma(\mathbf{s}))ds}$ can become very large, so large that $a^\varepsilon(\boldsymbol{\gamma}(\tau^\varepsilon)) = \frac{1}{\varepsilon}$ for some time $\tau_0 < \tau^\varepsilon < \infty$. At or before this time the trajectory must therefore have entered a “temporary” lake region $\{u^\varepsilon > g\}$ near 0. In this case, we have $0 \in \liminf_{\varepsilon \to 0} L^\varepsilon - L^0$.

On the other hand, if $\Delta g(0) < 0$, the exponential term becomes very small and the surface flow will not cause a temporary lake to form. Consequently part of the surface flow will be partly diverted to the left and partly to the right, as depicted in Figure 8. We provide as Figure 10 at the end of this paper some numerical simulations that support these conjectures.

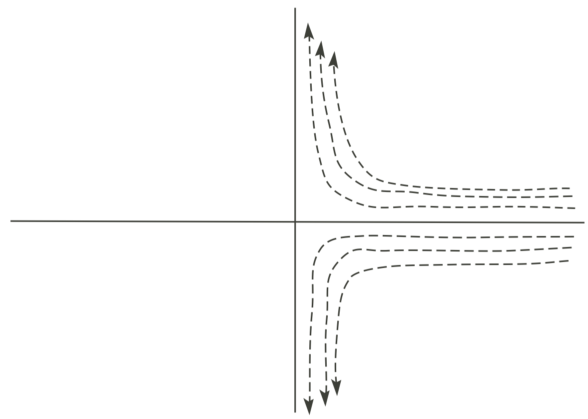

Figure 8: Diverted surface flow, $\Delta g < 0$.

6 Numerical studies

The figures appearing at the end of this paper were generated from MATLAB simulations of solutions to the time dependent "fast diffusion" PDE (1.1). Our numerics use straightforward finite-difference schemes, namely a centered-difference scheme to compute the spatial derivative and a forward-Euler scheme for the time derivative. Using these fully explicit methods, normally considered inferior to the implicit predictor-corrector schemes, is justified by the certain particular features of the PDE (1.1), namely (i) the integral in space of solutions is conserved, (ii) solutions never evolve below the height of the landscape g, and (iii) the term $a^\varepsilon \equiv 1$ in the interior of lake regions.

Implicit schemes achieve stability by virtue of their nonlocal behavior, which turns out to make them ill suited for our equation (1.1). Our computational experiments have shown that requiring solutions never drop below the landscape height seems to require a numerical method in which only the nearest neighbors $u^n_{k-1}, u^n_k, u^n_{k+1}$ influence u^{n+1}_k.

As for computing spatial second derivatives, we experimented with upwind-differencing schemes, like those used by Falcone-Finzi Vita in their papers [F-FV1], [F-FV2] on PDE related to (1.5) modeling sandpiles. Considering just one-dimensional equations, we seek a numerical method for evaluating the transport term $a_x u_x$ in the expansion $(au_x)_x = a_x u_x + au_{xx}$. The upwind-differencing method employs the right derivative u^+_x when a_x is positive, while the downwind scheme employs the left derivative u^-_x when a_x is negative. Consequently, we explored computing the transport term $a_x u_x$ by averaging the upwind and downwind methods with different weights, and equal weights seemed to work best. This is equivalent to using a centered-difference scheme to evaluate the divergence term $(au_x)_x$.

Furthermore, at points interior to lakes, we have $a^+_x = 0$ and $a^-_x = 0$; and so any combination of upwind and downwind methods will enforce our criterion (iii), namely that the term $a_x u_x$ vanishes.

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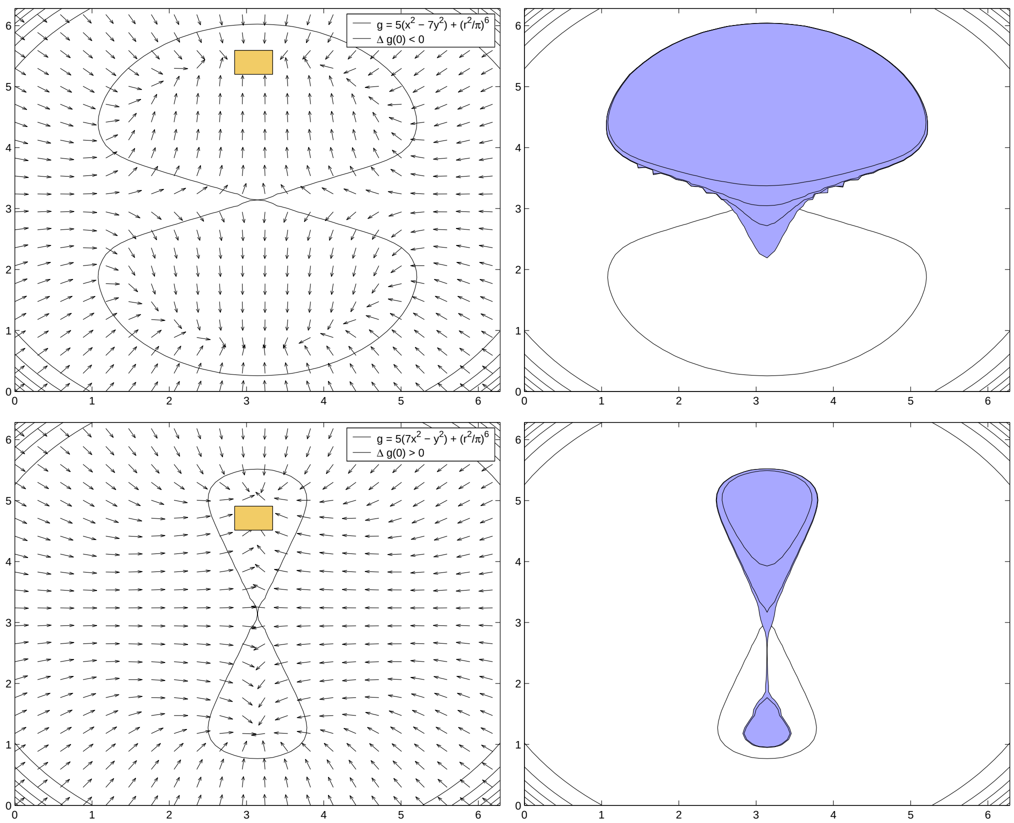

Figure 9: Overflowing lakes, with $\Delta g < 0$ (top row) and $\Delta g > 0$ (bottom row).

The shaded rectangles represent the support of the source term f and the downhill gradient flow lines determined by g are plotted. These pictures illustrate the conjectured differing behavior of the overflow at saddle points, depending upon the sign of Δg: see Section 5 for a discussion.

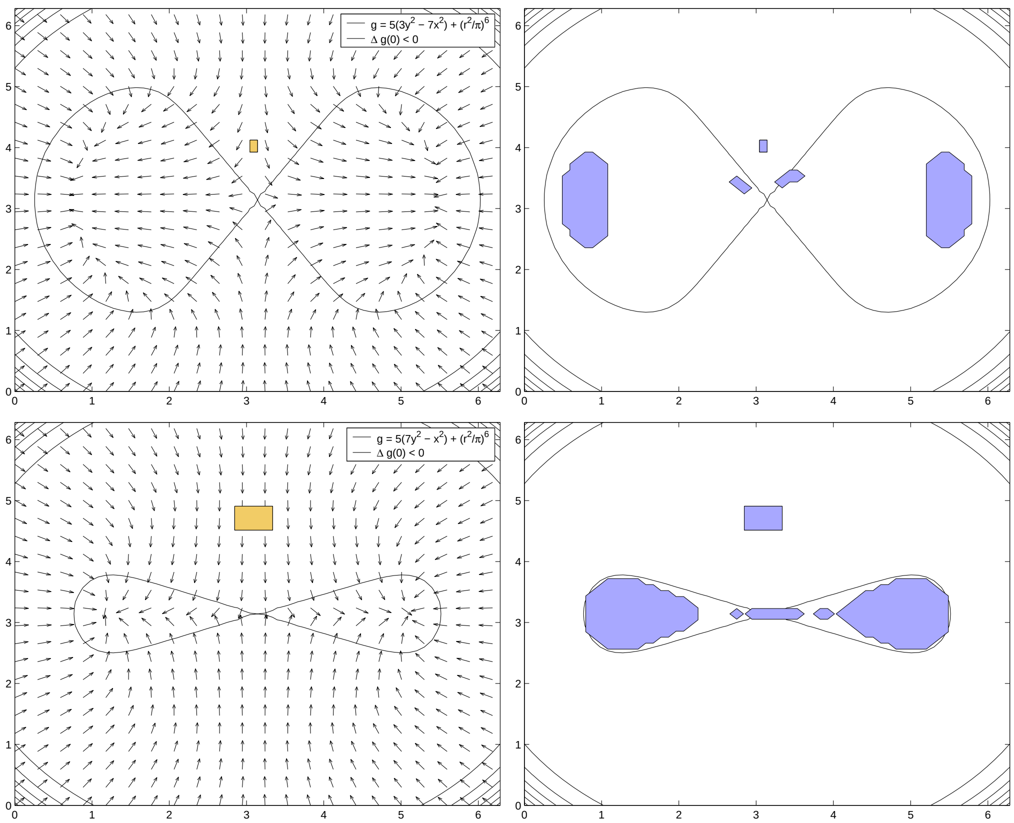

Figure 10: Flows into a saddle point, with $\Delta g < 0$ (top row) and $\Delta g > 0$ (top row).

Again shaded rectangles represents the support of the source term f, which in these pictures cause a downhill flow into a saddle point. As discussed in Section 5, we conjecture differing behavior at the saddle, depending upon the sign of Δg.