# IRREVERSIBILITY AND HYSTERESIS FOR A FORWARD–BACKWARD DIFFUSION EQUATION


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Abstract. Our intention in this paper is to publicize and extend somewhat important work of Plotnikov [P] on the asymptotic limits of solutions of viscous regularizations of an nonlinear diffusion PDE with a cubic nonlinearity. Since the formal limit PDE is in general ill–posed, we expect that the limit solves instead a corresponding diffusion equation with hysteresis effects. We employ entropy/entropy flux pairs to prove various assertions consistent with this expectation.


## 1. Introduction.

Initial value problems for nonlinear diffusion PDE of the general form

$$u_t = \Delta\phi(u) \tag{1.1}$$

are well–posed, and well–studied, provided the flux function $\phi : \mathbb{R} \to \mathbb{R}$ is nondecreasing. This paper considers instead smooth nonlinearities $\phi$ which violate this monotonicity condition, and have rather a cubic-type structure as illustrated below.

In this case the PDE (1.1) is ill–posed forwards in time whenever $u$ takes values in the instable "spinodal" interval $(b, a)$, where $\phi$ is decreasing and so (1.1) corresponds to a backward diffusion. Following Novick Cohen–Pego [NC-P] and Plotnikov [P], we replace (1.1) by the "viscous" regularization

$$u_t^\varepsilon = \Delta\phi(u^\varepsilon) + \varepsilon\Delta u_t^\varepsilon \tag{1.2}$$

and discuss the limit of $u^\varepsilon$ as $\varepsilon \to 0^+$.


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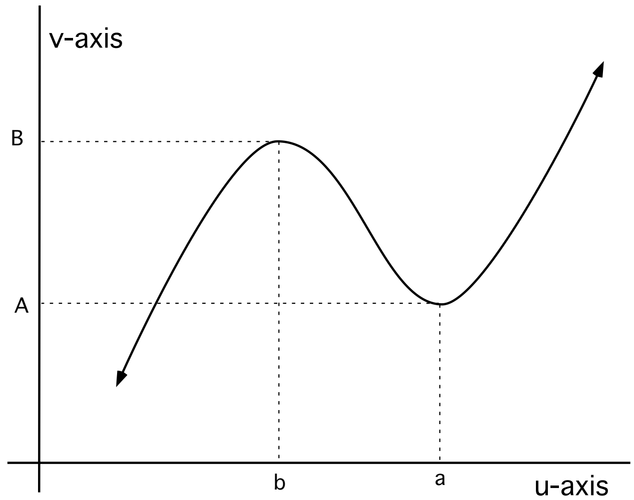



GRAPH OF $\phi$

We intend our paper to publicize this problem, to draw particular attention to Plotnikov [P], and to contribute some additional theoretical and numerical insights. Section 2 reviews the basic available estimates, including the basic "entropy" inequalities. In Section 3 we demonstrate that if a smooth boundary develops between the stable regions as $\epsilon \to 0$, then the entropy inequalities force this interface to move with direction changes predicted by a hysteresis loop built from the graph of $\phi$. The short Section 4 discusses a partial $L^1$ estimate for $u_t^\varepsilon$. We provide in Section 5 some numerical simulations, which support some of the heuristics developed earlier. The appendix discusses some measure theoretic tools for recording the irreversibility and hysteresis effects, and records an interesting, but unproved, formula $(A.7)$.

Our main new contribution is the analysis of the free boundary problem in Section 3. Our discussions elsewhere are, we think, interesting but not especially definitive. As we will see, it is remarkable that the PDE (1.2) admits the "entropy" type formulations (2.10); and it remains a fascinating problem to better exploit these to understand the limiting behavior as $\epsilon \to 0$.

Nishiura's book [N] is a good general reference for related issues, and in particular for other sorts of reaction–diffusion PDE governed by cubic–type nonlinearities. Brokate–Sprekels [B-S] and Visintin [V1, V2] provide much more information on hysteresis phenomena.

To keep this on-line version of our paper short, we omit a section on some prelimminary numerical simulations.

## 2. Estimates and weak convergence.

Take $U$ to be a smooth, bounded domain in $\mathbb{R}^n$, select a time $T > 0$, and let $\varepsilon > 0$. We turn our attention to the initial/boundary-value problem

$$\begin{cases} u_t^\varepsilon = \Delta\phi(u^\varepsilon) + \varepsilon\Delta u_t^\varepsilon & \text{in } U \times (0,T] \\ \frac{\partial}{\partial\nu}(\phi(u^\varepsilon) + \varepsilon u_t^\varepsilon) = 0 & \text{on } \partial U \times (0,T] \\ u^\varepsilon = u_0^\varepsilon & \text{on } U \times \{t=0\}, \end{cases} \tag{2.1}$$

$\nu$ denoting the unit outward normal to $\partial U$. The structure of this PDE is greatly clarified by introducing the new unknown function

$$v^\varepsilon := \phi(u^\varepsilon) + \varepsilon u_t^\varepsilon. \tag{2.2}$$

Then from (1.2) we have

$$\begin{cases} u_t^\varepsilon = \Delta v^\varepsilon \\ v^\varepsilon - \varepsilon\Delta v^\varepsilon = \phi(u^\varepsilon) \end{cases} \quad \text{in } U \times (0,T], \tag{2.3}$$

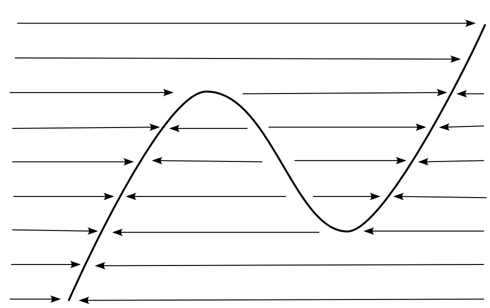
with the Neumann boundary condition

$$\frac{\partial v^\varepsilon}{\partial\nu} = 0 \quad \text{on } \partial U \times (0,T]. \tag{2.4}$$

Notice that (2.2) says

$$u_t^\varepsilon = \frac{v^\varepsilon - \phi(u^\varepsilon)}{\varepsilon} \tag{2.5}$$

Therefore for small $\varepsilon$, the time derivative $u_t^\varepsilon$ will be positive and large on the set $\{v^\varepsilon > \phi(u^\varepsilon)\}$, and negative and large on the set $\{v^\varepsilon < \phi(u^\varepsilon)\}$. So if we imagine the function $v^\varepsilon$ to be slowly–varying, the dynamics (2.5) should drive the system onto the stable part of the graph of $\phi$, where $\phi' \geq 0$. As the picture shows, these heuristics suggest the emergence of hysteresis effects in the small $\varepsilon$ limit.



Flow of the ODE, if v varies slowly

**Remark on the regularization.** A mathematical interpretation of the regularization (1.2) is this. Let $A$ denote the operator $-\Delta$, defined for functions with Neumann boundary conditions on $\partial U$. Then

$$J_\epsilon := (I + \epsilon A)^{-1}$$

is a form of the resolvent. The *Yosida approximation* of $A$ is

$$A_\varepsilon := AJ_\epsilon = \frac{I - J_\epsilon}{\varepsilon};$$

and the operator $A_\varepsilon$ is bounded, say on $L^2(U)$. According to (2.3), (2.4), we have $v_\varepsilon = J_\epsilon \phi(u^\varepsilon)$; and consequently

$$u_t^\varepsilon + A_\varepsilon \phi(u^\varepsilon) = 0. \tag{2.6}$$

In other words, our approximation (1.2) replaces the unbounded operator $A = -\Delta$ in the ill-posed evolution (1.1) with its Yosida approximation. Assuming $\phi$ is Lipschitz continuous, the operator $A_\varepsilon \phi(\cdot)$ is Lipschitz as well, and so the evolution (2.6) will have a unique solution, give the initial data. □

The really interesting question is understanding what happens to $u^\varepsilon$ and $v^\varepsilon = J_\epsilon \phi(u^\varepsilon)$, as $\varepsilon \to 0$.

**2.1. Estimates.** We assume the uniform bound

$$\sup_U |g^\varepsilon| \leq M \tag{2.5}$$

for a constant $M$ independent of $\varepsilon$. Using this, we easily establish

**Lemma 2.1.** *We have the estimates*

(i)

$$\sup_{U \times [0,T]} |u^\varepsilon, v^\varepsilon| \leq C_1, \tag{2.6}$$

(ii)

$$\int_0^T \int_U |Dv^\varepsilon|^2 + \varepsilon (u_t^\varepsilon)^2 \, dxdt \leq C_2 \tag{2.7}$$

*for constants $C_1$, $C_2$ depending only on $M$, $\phi$ and $n$.*

We next generalize estimate (2.6) as follows, closely following [NC-P] and [P]. Take

$$g : \mathbb{R} \to \mathbb{R} \text{ to be nondecreasing,} \tag{2.8}$$

and set

$$G(z) := \int_0^z g(\phi(s))\,ds + C \tag{2.9}$$

for any constant $C$. Thus $G'(z) = g(\phi(z))$. If $g$ is smooth, we compute from (2.3) that

$$G(u^\varepsilon)_t = \operatorname{div}(g(v^\varepsilon)Dv^\varepsilon) - g'(v^\varepsilon)|Dv^\varepsilon|^2 - (g(v^\varepsilon) - g(\phi(u^\varepsilon)))\left(\frac{v^\varepsilon - \phi(u^\varepsilon)}{\varepsilon}\right). \tag{2.10}$$

The key observation is that the last two terms are nonnegative, and so this calculation is strongly reminiscent of "entropy/entropy flux" calculations for dissipative approximations to conservation laws (cf. [E]).

We obtain upon integrating

**Lemma 2.2.** *For each smooth, nondecreasing function $g$,*

$$\int_0^T \int_U g'(v^\varepsilon)|Dv^\varepsilon|^2 + \mu_g^\varepsilon \,dxdt \leq C_3 \tag{2.11}$$

*where*

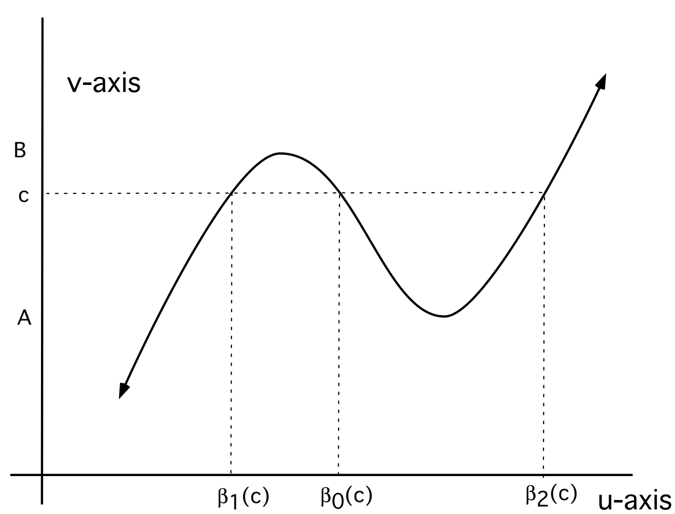
$$\mu_g^\varepsilon := (g(v^\varepsilon) - g(\phi(u^\varepsilon)))\left(\frac{v^\varepsilon - \phi(u^\varepsilon)}{\varepsilon}\right) \geq 0 \tag{2.12}$$

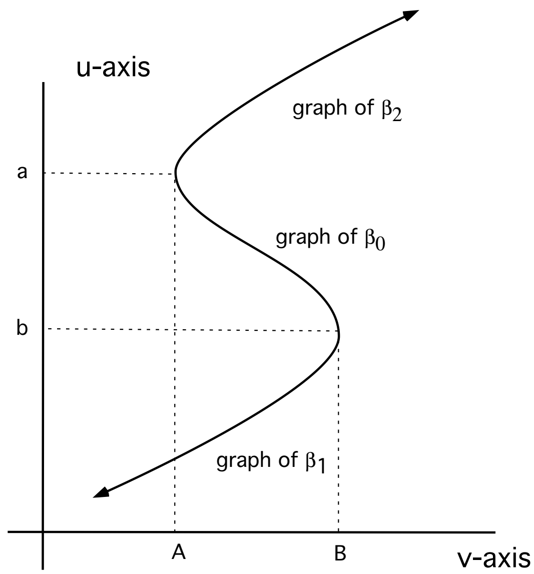
*and $C_3$ is a constant depending only on $M$ and $\|G\|_{L^\infty}$.*

**2.2. Weak convergence.** In view of these estimates, there exists a sequence $\varepsilon_j \to 0$ and bounded functions $u, v$ such that

$$\begin{cases} u^{\varepsilon_j} \rightharpoonup u & \text{weakly} * \text{ in } L^\infty(U \times (0,T)) \\ v^{\varepsilon_j} \rightharpoonup v & \text{weakly} * \text{ in } L^\infty(U \times (0,T)) \\ Dv^{\varepsilon_j} \rightharpoonup Dv & \text{weakly in } L^2(U \times (0,T)). \end{cases} \tag{2.13}$$

Our main goal is understanding the relationships between $u, v$ and the equations they satisfy. Plotnikov [P] has deeply studied this issue, coming to the following conclusions. First, let us as illustrated introduce the three branches $\beta_i$ $(i = 0, 1, 2)$ of $\phi^{-1}$:






GRAPH OF THE INVERSE FUNCTIONS

**Theorem 2.1** ([P]). *There exist three measurable functions $\lambda_0, \lambda_1, \lambda_2$, such that for a.e. point $(x,t) \in U \times (0,T]$ we have*

(i) $0 \le \lambda_i \le 1 \quad (i = 0,1,2)$

(ii) $\sum_{i=0}^{2} \lambda_i = 1$.

(iii) *Furthermore, passing as necessary to a further subsequence,*

$$F(u^{\varepsilon_j}) \rightharpoonup \bar{F} := \sum_{i=0}^{2} \lambda_i F(\beta_i(v)) \tag{2.14}$$

*weakly $*$ in $L^\infty(U \times (0,T))$ for each continuous function $F$.*

(iv) *In addition,*

$$v^{\varepsilon_j}, \phi(u^{\varepsilon_j}) \to v \qquad \text{strongly in } L^2(U \times (0,T)). \tag{2.15}$$

We call $\lambda_0, \lambda_1, \lambda_2$ the *phase fractions*. The importance of assertion (2.14) is its characterization of the limiting behavior of the $u^{\varepsilon_j}$. Very roughly speaking, this possibly highly oscillating sequence takes the fraction $\lambda_i$ of its values near the branch $u = \beta_i(v)$, for $i = 0,1,2$.

Next we take a smooth, nonnegative function $\zeta \in C_c^\infty(U \times [0,T])$, multiply (2.10) by $\zeta$ and integrate by parts:

$$\int_0^T \int_U -G(u^\varepsilon)\zeta_t + g(v^\varepsilon) Dv^\varepsilon \cdot D\zeta \, dxdy = \int_0^T \int_U (g'(v^\varepsilon)|Dv^\varepsilon|^2 + \mu_g^\varepsilon)\zeta \, dxdt.$$

Passing to limits as $\varepsilon = \varepsilon_j \to 0$ and recalling (2.14), (2.15) we conclude that

$$\bar{G}_t - \operatorname{div}(g(v)Dv) \le -g'(v)|Dv|^2 \quad \text{in } U \times (0,T) \tag{2.16}$$

for each nondecreasing function $g$ as above. Similarly,

$$u_t = \Delta v \quad \text{in } U \times (0,T). \tag{2.17}$$

**Remark: failure of strong convergence.** It is certainly possible to arrange the initial data so that any values of the phase fractions $\lambda_i$ $(i = 0,1,2)$ can occur at a time $t > 0$, subject only to the constraint that their sum be one.

To see this, first fix a value $A < c < B$. Then select for $\varepsilon > 0$ an intial function $g^\varepsilon$ having the form:

$$u_0^\varepsilon := \begin{cases} \beta_0(c) & \text{on } E_0^\varepsilon \\ \beta_1(c) & \text{on } E_1^\varepsilon \\ \beta_2(c) & \text{on } E_2^\varepsilon, \end{cases}$$

where $E_0^\varepsilon, E_1^\varepsilon, E_2^\varepsilon$ are arbitrary measurable, disjoint sets, whose union is $U$. Since $\phi(u_0^\varepsilon) \equiv c$, the solution of (2.1) does not depend on time: $u^\varepsilon \equiv u_0^\varepsilon$.

Given any three nonnegative measurable functions $\lambda_0, \lambda_1, \lambda_2$, whose sum is identically 1, we can construct the sets $E_0^\varepsilon, E_1^\varepsilon, E_2^\varepsilon$ so that $\lambda_0, \lambda_1, \lambda_2$ are the resulting phase fractions, in the sense of Theorem 2.1. In this example the functions $\lambda_i$ do not depend on the time variable $t$, but are essentially arbitrary in the $x$-variables. □

## 3. A free boundary problem with hysteresis.

In this section we illustrate the utility of the differential inequalities (2.16) if the phase fractions $\lambda_0, \lambda_1, \lambda_2$ have a particularly simple structure. We therefore assume that

$$\lambda_0 = 0 \quad \text{a.e. in } U \times [0,T] \tag{3.1}$$

and

$$\begin{cases} \lambda_1 = 1 & \text{a.e. in } V_1 \\ \lambda_2 = 1 & \text{a.e. in } V_2, \end{cases}$$

where $V_1, V_2$ are two open regions in $U \times (0,T)$, with a smooth, $n$-dimensional interface

$$\Gamma := \bar{V}_1 \cap \bar{V}_2.$$

In other words, we are supposing that

$$\begin{cases} u^{\varepsilon_j} \to \beta_1(v) & \text{a.e. in } V_1 \\ u^{\varepsilon_j} \to \beta_2(v) & \text{a.e. in } V_2, \end{cases}$$

and that there is a smooth free boundary $\Gamma$ separating the two pure phase regions $V_1, V_2$.

We want to deduce the behavior of $u$ in $V_1, V_2$ and to understand as well how the interface $\Gamma$ moves. We suppose that $u, v$ are smooth in $\bar{V}_1, \bar{V}_2$. For each point on $\Gamma$, let

$$\bar{\nu} = (\nu^1, \ldots, \nu^n, \nu^{n+1}) = (\nu, \nu^{n+1})$$

denote the unit normal in $\mathbb{R}^{n+1}$ pointing into $V_1$. Let $u_1, v_1$ denote the values along $\Gamma$ from within $V^1$ and $u_2, v_2$ the values along $\Gamma$ from within $V^2$.

**Theorem 3.1.** (i) *We have*

$$\begin{cases} \beta_1(v)_t = \Delta v & \text{in } V_1 \\ \beta_2(v)_t = \Delta v & \text{in } V_2. \end{cases} \tag{3.2}$$

(ii) *Furthermore,*

$$v_1 = v_2 \quad \text{along } \Gamma \tag{3.3}$$

*and*

$$\nu^{n+1}[u] = \nu \cdot [D_x v] \quad \text{along } \Gamma, \tag{3.4}$$

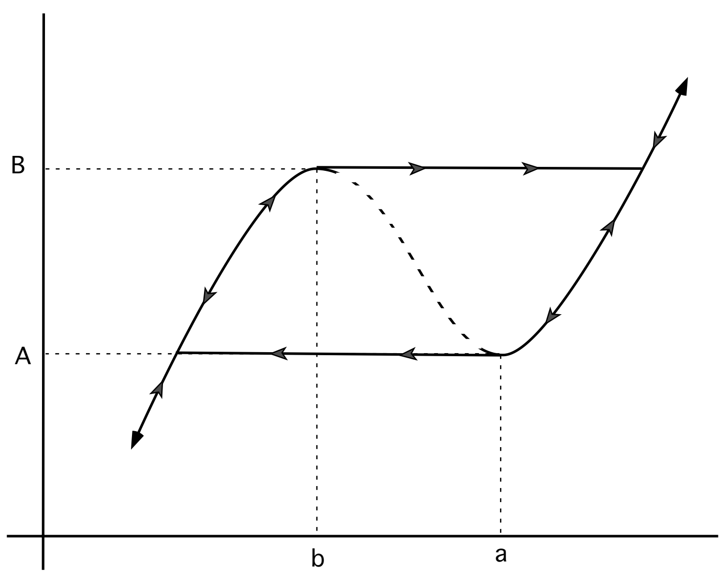
*where* $[\cdot]$ *denotes a jump across the interface. That is,*

$$[u] := u_1 - u_2, \; [D_x u] := D_x v_1 - D_x v_2.$$

(iii) *Also,*

$$\begin{cases} \nu^{n+1} = 0 & \text{if } v \neq A, B \\ \nu^{n+1} \geq 0 & \text{if } v = A \\ \nu^{n+1} \leq 0 & \text{if } v = B, \end{cases} \tag{3.5}$$

*where we write* $v = v^1 = v^2$ *along* $\Gamma$.




A HYSTERESIS LOOP

**Interpretation.** According to (3.5), the interface $\Gamma$ moves, which is to say that a phase transition occurs, only if $v = A$ or $B$. Furthermore, if $(x_0, t_0) \in \Gamma$, the surface moves so that for some small $\varepsilon > 0$, $(x_0, t_0 - \varepsilon)$ lies in phase 1 and $(x_0, t_0 + \varepsilon)$ lies in phase 2 only if $v(x_0, t_0) \approx B$. Likewise $(x_0, t_0 - \varepsilon)$ lies in phase 2 and $(x_0, t_0 + \varepsilon)$ lies in phase 1 only if $v(x_0, t_0) \approx A$.

We can therefore envision the phase transitions as tracing out a clockwise hysteresis loop, as illustrated. □

*Proof.* 1. We have

$$\bar{G} = \begin{cases} G(\beta_1(v)) & \text{in } V_1 \\ G(\beta_2(v)) & \text{in } V_2, \end{cases} \tag{3.6}$$

for each function $G$ as above. In particular,

$$u = \begin{cases} \beta_1(v) & \text{in } V_1 \\ \beta_2(v) & \text{in } V_2; \end{cases} \tag{3.7}$$

and so (3.2) follows from (2.18). Also, (2.18) implies

$$\begin{aligned} 0 &= \int_0^T \int_U -u\zeta_t + Dv \cdot D\zeta \, dxdt \\ &= \iint_{V_1} -\beta_1(v)\zeta_t + Dv \cdot D\zeta \, dxdt + \iint_{V_2} -\beta_2(v)\zeta_t + Dv \cdot D\zeta \, dxdt \end{aligned}$$

for each $\zeta \in C_c^\infty$. Integrating by parts and remembering (3.2), (3.7), we deduce

$$0 = \int_\Gamma (\nu^{n+1}[u] - \nu \cdot [D_x v])\zeta \, d\mathcal{H}^n.$$

This identity implies the Rankine–Hugoniot relation (3.4).

2. We multiply (2.17) by a nonnegative function $\zeta \in C_c^\infty$ and integrate by parts, to find

$$\begin{aligned} 0 &\geq \int_0^T \int_U -\bar{G}\zeta_t + g(v)Dv \cdot D\zeta + g'(v)|Dv|^2\zeta \, dxdt \\ &= \iint_{V_1} -G(\beta_1(v))\zeta_t + g(v)Dv \cdot D\zeta + g'(v)|Dv|^2\zeta \, dxdt \\ &\quad + \iint_{V_2} -G(\beta_2(v))\zeta_t + g(v)Dv \cdot D\zeta + g'(v)|Dv|^2\zeta \, dxdt. \end{aligned}$$

We once more integrate by parts, remembering that

$$G'(\beta_1(v)) = g(\phi(\beta_1(v))) = g(v), \; G'(\beta_2(v)) = g(v).$$

It follows that

$$0 \geq \iint_{V_1} g(v)(b_1(v)_t - \Delta v)\zeta \, dxdt$$
$$+ \iint_{V_2} g(v)(\beta_2(v)_t - \Delta v)\zeta \, dxdt + \int_\Gamma (\nu^{n+1}[G(u)] - \nu \cdot [D_x v] g(v))\zeta \, d\mathcal{H}^n,$$

and consequently

$$\nu^{n+1}[G(u)] - \nu \cdot [D_x v] g(v) \leq 0 \quad \text{along } \Gamma.$$

Substituting (3.4), we rewrite this inequality to read

$$\nu^{n+1}([G(u)] - g(v)[u]) \leq 0 \quad \text{along } \Gamma$$

for each nondecreasing function $g$.

Since $G'(z) = g(\phi(z))$, we can recast this into the form

$$\nu^{n+1} \left( \int_{\beta_1(v)}^{\beta_2(v)} g(\phi(s)) - g(v) \, ds \right) \geq 0 \quad \text{along } \Gamma. \tag{3.8}$$

3. Clearly $A \leq v \leq B$ along $\Gamma$. If $A < v < B$, we first take $g^+$ to be zero on $(-\infty, v]$, positive and nondecreasing on $(v, \infty)$. Then

$$\int_{\beta_1(v)}^{\beta_2(v)} g^+(\phi(s)) - g^+(v) \, ds > 0$$

and so $\nu^{n+1} \geq 0$. Next select $g^-$ to be negative and nondecreasing on $(-\infty, v)$, zero on $[v, \infty)$. This forces

$$\int_{\beta_1(v)}^{\beta_2(v)} g^-(\phi(s)) - g^-(v) \, ds < 0;$$

whence $\nu^{n+1} \leq 0$. Consequently $\nu^{n+1} = 0$ if $A < v < B$.

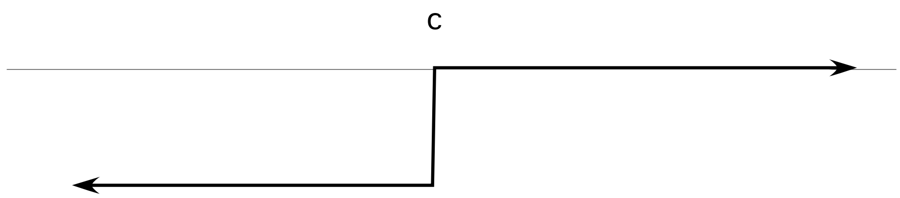
If $v = A$, we take $g^+$ as above, to deduce $\nu^{n+1} \geq 0$. Likewise, $\nu^{n+1} \leq 0$ if $v = B$. □

## 4. Entropies built from step functions, a partial $\mathbf{L^1}$ estimate of $\mathbf{u_t^\epsilon}$.

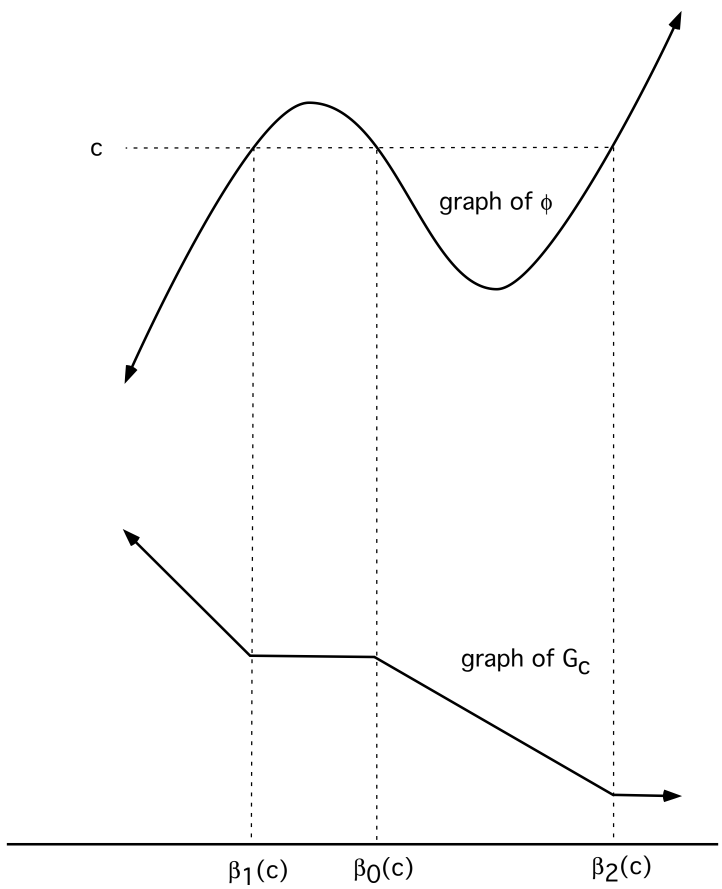
We can squeeze out a bit more information by taking a particularly simple choice for the function $g$.

Fix $c \in \mathbb{R}$ and set

$$g_c(z) := \begin{cases} -1 & \text{if } z < c \\ 0 & \text{if } z > c. \end{cases}$$




Graph of $g_c$




Then for $A < c < B$, we have

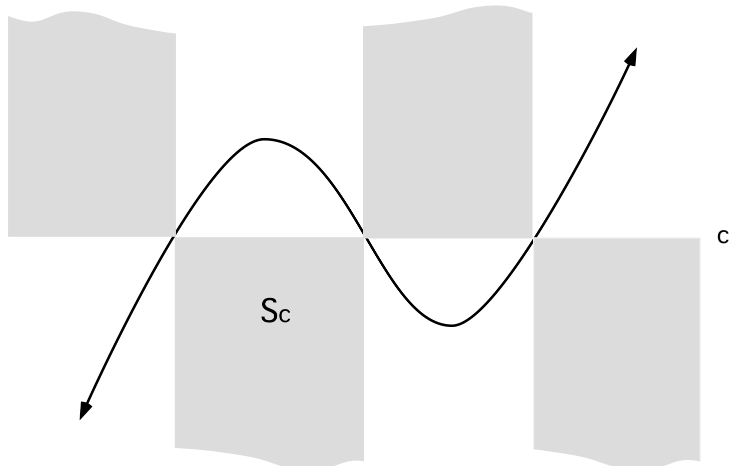
$$G_c'(z) = g_c(\phi(z)) = \begin{cases} -1 & \text{if } z < \beta_1(c) \\ 0 & \text{if } \beta_1(c) < z < \beta_0(c) \\ -1 & \text{if } \beta_0(c) < z < \beta_2(c) \\ 0 & \text{if } z > \beta_2(c). \end{cases}$$

According to (2.11), (2.12), we have

$$\int_0^T \int_U \mu_{g_c}^{\varepsilon} \, dxdt \le C$$

for

$$\mu_{g_c}^{\varepsilon} := (g_c(v^\varepsilon) - g_c(\phi(u^\varepsilon))) \left( \frac{v^\varepsilon - \phi(u^\varepsilon)}{\varepsilon} \right).$$




Recalling the definition of the step function $g_c$, we deduce this "partial" $L^1$ estimate on $u_t^\varepsilon$:

$$\iint_{\{(u^\varepsilon, v^\varepsilon) \in S_c\}} |u_t^\varepsilon| \, dxdt \leq C,$$

for

$$S_c := \{(x,t) \,|\, \phi((u^\varepsilon) \leq c \leq v^\varepsilon \text{ or } v^\varepsilon \leq c \leq \phi((u^\varepsilon)\}.$$

See the picture. Likewise

$$\iint_{\{|v^\varepsilon - \phi(u^\varepsilon)| \geq \varepsilon\}} |u_t^\varepsilon| \, dxdt \leq C.$$

**Remark: estimates on time derivatives.** It remains an outstanding problem to improve these $L^1$ estimates on the time derivatives $u_t^\varepsilon$. There is some hope that such bounds may be valid, since according to Little–Showalter [L-S] and Visintin [V1, V2], certain related hysteresis/diffusion models correspond to flows which are contractions in the $L^1$ norm. □

## Appendix : Measures of irreversibility.

We introduce some measures which record certain irreversibility phenomena, and also state an interesting, but unproved, formula relating these measures and the time derivatives of the phase fractions.

**Theorem A.1.** (i) *Let $\Phi$ denote an antiderivative of $\phi$. There exist nonnegative Radon measures $\rho, \mu$ on $\bar{U} \times [0,T]$ such that*

$$(A.1) \qquad \begin{cases} |Dv^{\varepsilon_j} - Dv|^2 \rightharpoonup \rho \\ \varepsilon_j (u_t^{\varepsilon_j})^2 \rightharpoonup \mu \end{cases}$$

*weakly as measures. Also*

$$(A.2) \qquad \bar{\Phi}_t - \operatorname{div}(vDv) = -|Dv|^2 - \nu$$

*in $U \times (0,\infty)$, for*

$$\nu := \rho + \mu.$$

(ii) *For each nondecreasing $C^1$ function $g$, there exist nonnegative Radon measures $\rho_g, \mu_g$ such that*

$$(A.3) \qquad \begin{cases} g'(v^{\varepsilon_j})|Dv^{\varepsilon_j} - Dv|^2 \rightharpoonup \rho_g \\ \mu_g^{\varepsilon_j} := (g(v^{\varepsilon_j}) - g(\phi(u^{\varepsilon_j}))) \left(\frac{v^{\varepsilon_j} - \phi(u^{\varepsilon_j})}{\varepsilon_j}\right) \rightharpoonup \mu_g. \end{cases}$$

*Furthermore*

$$(A.4) \qquad \bar{G}_t - \operatorname{div}(g(v)Dv) = -g'(v)|Dv|^2 - \nu_g$$

*in $U \times (0,T)$, for*

$$\nu_g := \rho_g + \mu_g.$$

We can formally rewrite $(A.2)$, $(A.4)$ in the form

$$(A.5) \qquad \begin{cases} \bar{\Phi}_t - v\Delta v = -\nu \\ \bar{G}_t - g(v)\Delta v = -\nu_g. \end{cases}$$

□

*Proof.* 1. According to estimate (2.7), we have

$$\int_0^T \int_U |Dv^\varepsilon - Dv|^2 + \varepsilon (u_t^\varepsilon)^2 \, dxdt \le C$$

for some constant $C_1$ independent of $\varepsilon$. Passing if necessary to a subsequence, we have $(A.1)$. Consider next the PDE

$$\Phi(u^\varepsilon)_t - \operatorname{div}(v^\varepsilon Dv^\varepsilon) = -|Dv^\varepsilon|^2 - \varepsilon(u_t^\varepsilon)^2. \tag{A.6}$$

We write

$$-|Dv^\varepsilon|^2 = |Dv|^2 - 2Dv \cdot Dv^\varepsilon - |Dv^\varepsilon - Dv|^2,$$

to see that

$$|Dv^{\varepsilon_j}|^2 \rightharpoonup |Dv|^2 + \rho.$$

We pass to limits in $(A.6)$ as $\varepsilon = \varepsilon_j \to 0$, discovering

$$\bar{\Phi}_t - \operatorname{div}(vDv) = -|Dv|^2 - \rho - \mu.$$

2. Likewise we recall the estimate (2.11) to deduce, upon passing if necessary to a further subsequence, that $(A.3)$ holds. The formula $(A.4)$ results from the identity

$$G(u^\varepsilon)_t - \operatorname{div}(g(v^\varepsilon)Dv^\varepsilon) = -g'(v^\varepsilon)|Dv^\varepsilon|^2 - \mu_g^\varepsilon.$$

□

**A representation formula.** We record next some formal calculations using the entropies that suggest the following interesting, but unproved, formula. For any open set on which $\{A < v < B\}$ and any piecewise $C^1$, nondecreasing function $g$, we have

$$\begin{aligned} \nu_g &= \lambda_{1,t} \int_{\beta_1(v)}^{\beta_0(v)} g(\phi(r)) - g(v)\, dr \\ &\quad + \lambda_{2,t} \int_{\beta_0(v)}^{\beta_2(v)} g(v) - g(\phi(r))\, dr \\ &= \lambda_{1,t} \int_v^B g'(s)(\beta_0(s) - \beta_1(s))\, ds \\ &\quad + \lambda_{2,t} \int_A^v g'(s)(\beta_2(s) - \beta_0(s))\, ds. \end{aligned} \tag{A.7}$$

A formal derivation of $(A.7)$ is this. According to $(A.5)$ and (2.17),

$$\nu_g = -\bar{G}_t + g(v)\Delta v = -\bar{G}_t + g(v)u_t.$$

But

$$u = \sum_{i=0}^{2} \lambda_i \beta_i(v), \quad \bar{G} = \sum_{i=0}^{2} \lambda_i G(\beta_i(v));$$

and therefore

$$\nu_g = -\sum_{i=0}^{2} \left(\lambda_{i,t} G(\beta_i(v)) + \lambda_{i,t} G'(\beta_i(v))\beta_i(v)_t\right) + g(v) \sum_{i=0}^{2} \left(\lambda_{i,t}\beta_i(v) + \lambda_i \beta_i(v)_t\right).$$

Since $G'(\beta_i(v)) = g(\phi(\beta_i(v))) = g(v)$, we deduce

$$\nu_g = \sum_{i=0}^{2} \lambda_{i,t}(g(v)\beta_i(v) - G(\beta_i(v))).$$

But $\lambda_{0,t} = -\lambda_{1,t} - \lambda_{2,t}$, and so we can simplify the foregoing to read

$$\begin{aligned}\nu_g = \lambda_{1,t}&[g(v)(\beta_1(v) - \beta_0(v)) - [G(\beta_1(v)) - G(\beta_0(v))]\\ &+ \lambda_{2,t}[g(v)(\beta_2(v) - \beta_0(v)) - [G(\beta_2(v)) - G(\beta_0(v))].\end{aligned}$$

This expression is equivalent to $(A.7)$.

The identity $(A.7)$ is suggestive, in that it relates the time derivatives of the phase fractions $\lambda_1, \lambda_2$ and the dissipation measure $\nu_g$, within the region $\{A < v < B\}$. But it is not so clear to us how to make the derivation rigorous.

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