Further PDE Methods for Weak KAM Theory

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Abstract

We introduce and make estimates for several new approximations that in appropriate asymptotic limits yield the key PDE for weak KAM theory, namely a Hamilton-Jacobi type equation for a potential u and a coupled transport equation for a measure σ.

We revisit as well a singular variational approximation introduced in [E1], and demonstrate "approximate integrability" of certain phase space dynamics related to the Hamiltonian flow. Other examples include a pair of strongly coupled PDE suggested by the Lions-Lasry theory [L-L1] of mean field games and a new and extremely singular elliptic equation suggested by sup-norm variational theory.

1 Introduction

The PDE approach to weak KAM theory (see Fathi [F4] or [E3]) focusses upon two fundamental equations, the Hamilton-Jacobi type equation

$$H(Du, x) = \bar{H}(P) \tag{1.1}$$

and the coupled transport (or continuity) equation

$$\operatorname{div}(D_p H(Du, x)\sigma) = 0, \tag{1.2}$$

which is the adjoint of the linearization of (1.1). Here the Hamiltonian $H = H(p, x)$ is assumed to be nonnegative, uniformly convex in p and $\mathbb{T}^n$-periodic in x, $\mathbb{T}^n$ denoting the unit cube in $\mathbb{R}^n$ with opposite faces identified. We introduce the vector $P \in \mathbb{R}^n$ for reasons

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that will be apparent later. The unknowns for (1.1) are both the value of the *effective Hamiltonian* $\bar{H}$ at P and the potential $u = P \cdot x + v$, where v is $\mathbb{T}^n$-periodic in x. The unknown in (1.2) is the measure σ.

The overall goals are (i) to ascertain the solvability of the PDE (1.1) and (1.2), and, more importantly, (ii) to understand what these two equations imply concerning the Hamiltonian dynamics

$$\begin{cases} \dot{\mathbf{x}} = D_p H(\mathbf{p}, \mathbf{x}) \\ \dot{\mathbf{p}} = -D_x H(\mathbf{p}, \mathbf{x}). \end{cases} \tag{1.3}$$

This is a continuation and elaboration of my earlier papers [E1] and [E2], which introduced two unusual PDE approximations into weak KAM theory. We are particularly interested in finding smooth approximations to the equations (1.1) and (1.2), for which various formal calculations giving information about the dynamics (1.3) can be made rigorous.

In Section 2 we revisit the singular variational scheme proposed in [E1], and deduce from the estimates found there a sort of "approximate integrability" assertion for certain trajectories of (1.3). Sections 3 and 4 propose two completely new approximation schemes, one Hamiltonian the other Lagrangian, and both involving the Donsker-Varadhan I functional [D-V]. We show in both cases how analogs of (1.1), (1.2) arise, and derive some basic estimates. Section 5 points out that the recent mean field game theoretical methods of Lions and Lasry [L-L1] yield in the deterministic case a strongly coupled pair of equations generalizing (1.1), (1.2). We modify some of our estimates, showing that in certain cases the strong coupling in fact regularizes the measure σ. In Section 6 we propose a highly speculative "second order" variant of the PDE for weak KAM theory and derive a few estimates.

Hypotheses on $\boldsymbol{H}$. We suppose throughout that the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $H = H(p, x)$, is smooth, nonnegative, and satisfies these conditions:

(i) For each $p \in \mathbb{R}^n$, the mapping $x \mapsto H(p, x)$ is $\mathbb{T}^n$-periodic.

(ii) There exists a constant $\gamma > 0$ such that

$$H_{p_i p_j}(p, x)\xi_i \xi_j \geq \gamma |\xi|^2 \tag{1.4}$$

for all $p, x, \xi \in \mathbb{R}^n$.

(iii) There exists a constant C such that

$$|D_p^2 H(p, x)| \leq C, \ |D_{xp}^2 H(p, x)| \leq C(1 + |p|), \ |D_x^2 H(p, x)| \leq C(1 + |p|^2)$$

for all $p, x \in \mathbb{R}^n$.

2 Entropy regularization, approximate integrability

In this section we return to, and reinterpret, the singular variational problem introduced in [E1] and also explain how minimizers $v_k = v_k(P, x)$ of the functional

$$I_k[v] := \int_{\mathbb{T}^n} e^{kH(P+Dv,x)}\, dx \tag{2.1}$$

in the singular limit $k \to \infty$ provide some information about the Hamiltonian dynamics (1.3). This section should be regarded as an addendum to [E1].

(M. Rorro [R] has developed effective numerical methods for computing both the effective Hamiltonian and the measure σ, using the variational approximation (2.1).)

2.1 Entropy and approximation. We can interpret our approximation (2.1) by introducing the entropy $h(\mu)$ of a Borel probability measure μ on $\mathbb{T}^n$, defined as

$$h(\mu) := \begin{cases} \int_{\mathbb{T}^n} f \log f \, dx & \text{if } d\mu = f dx \\ +\infty & \text{otherwise.} \end{cases}$$

Then for each fixed function v, we have

$$\sup_{\mu} \left\{ \int_{\mathbb{T}^n} H(P+Dv,x)\, d\mu - \frac{1}{k} h(\mu) \right\} = \frac{1}{k} \log \left(\int_{\mathbb{T}^n} e^{kH(P+Dv,x)}\, dx \right) = \frac{1}{k} \log I_k[v], \tag{2.2}$$

the supremum attained at the measure

$$d\mu = \frac{e^{kH(P+Dv,x)}}{Z} dx,$$

normalized by $Z := \int_{\mathbb{T}^n} e^{kH(P+Dv,x)} dx$.

We will in Sections 3 and 4 below introduce various alternatives to the functional (2.2).

2.2 Hamiltonian viewpoint. For the reader's convenience, we briefly review the connection between the variational integrand (2.1) and the Hamiltonian dynamics (1.3). Setting

$$u^k := P \cdot x + v^k,$$

we note first that the Euler–Lagrange equation associated with (2.1) reads

$$\operatorname{div}(e^{kH} D_p H) = 0, \tag{2.3}$$

H evaluated at $H = H(Du^k, x)$. Writing

$$\bar{H}^k(P) := \frac{1}{k} \log \left(\int_{\mathbb{T}^n} e^{kH(Du^k,x)}\, dx \right), \quad \sigma^k := e^{k(H(Du^k,x) - \bar{H}^k(P))}, \tag{2.4}$$

we have

$$\operatorname{div}(\sigma^k D_p H) = 0,\ \ \sigma^k \geq 0,\ \ \int_{\mathbb{T}^n} \sigma^k \, dx = 1.$$

In particular,

$$\bar{H}_k(P) = \inf_v \sup_\mu \left\{ \int_{\mathbb{T}^n} H(P + Dv, x) \, d\mu - \frac{1}{k} h(\mu) \right\}.$$

The function $u^k = u^k(P, x)$ is smooth. Also, estimates derived in [E1] provide the uniform bounds

$$\max_{\mathbb{T}^n} |u^k|, |Du^k| \leq C.$$

Sending $k \to \infty$, we may assume, passing if necessary to subsequence, that $u^k \to u = P \cdot x + v$ uniformly on $\mathbb{T}^n$ and $\sigma^k dx \rightharpoonup d\sigma$ weakly as measures on $\mathbb{T}^n$. I proved in [E1] that $\lim_{k\to\infty} \bar{H}^k(P) = \bar{H}(P)$ and

$$\lim_{k\to\infty} \int_{\mathbb{T}^n} H(Du^k, x)\sigma^k \, dx = \bar{H}(P), \tag{2.5}$$

where $\bar{H} = \bar{H}(P)$ is the *effective Hamiltonian* associated with $H = H(p, x)$, introduced by Lions, Papanicolaou, and Varadhan [L-P-V]. Furthermore,

$$H(Du, x) \leq \bar{H}(P) \text{ a.e.};$$

u is almost everywhere differentiable on the support of σ;

$$H(Du, x) = \bar{H} \ \ \sigma\text{-a.e.}; \tag{2.6}$$

and

$$\operatorname{div}(\sigma D_p H(Du, x)) = 0. \tag{2.7}$$

Equation (2.6) is a version of our basic Hamilton-Jacobi PDE (1.1), and (2.7) is the transport PDE (1.2).

2.3 Lagrangian viewpoint. The *Lagrangian* $L = L(v, x)$ associated with H is

$$L(v, x) := \max_p (p \cdot v - H(p, x)).$$

Passing as necessary to a subsequence, we may assume for all continuous functions $\Phi = \Phi(p, x)$ that

$$\lim_{k\to\infty} \int_{\mathbb{T}^n} \Phi(Du^k, x)\sigma^k \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{T}^n} \Phi(p, x) \, d\nu \tag{2.8}$$

for some probability measure ν on the cotangent bundle $\mathbb{R}^n \times \mathbb{T}^n$. We then use the change of variables $v = D_pH(p,x), p = D_vL(v,x)$ to push ν to a probability measure μ on the tangent bundle, defined by the formula

$$\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} \Psi(v,x)\,d\mu = \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} \Psi(D_pH(p,x),x)\,d\nu \tag{2.9}$$

for all continuous $\Psi = \Psi(v,x)$.

Define

$$V := \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} v\,d\mu. \tag{2.10}$$

I proved in [E1] that μ is a minimizer of Mather's action functional

$$A[\mu] := \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} L(v,x)\,d\mu \tag{2.11}$$

among all probability measures on $\mathbb{R}^n \times \mathbb{T}^n$, satisfying the constraint (2.10) and the flow invariance condition

$$\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} v \cdot D\phi\,d\mu = 0 \tag{2.12}$$

for all smooth and $\mathbb{T}^n$-periodic $\phi = \phi(x)$. The minimum value is $\bar{L}(V)$, where the *effective Lagrangian* $\bar{L}$ is the convex dual of $\bar{H}$.

Later in this paper we will introduce several other quite different approximations that also yield probability measures μ minimizing (2.11), subject to (2.10) and (2.12).

2.4 Canonical change of variables. This section is motivated by the classical observation (see for instance [E3]) that if $u = u(P,x)$ is a smooth solution of (1.1) and if we can solve the expressions

$$\begin{cases} p = D_xu(P,x) \\ X = D_Pu(P,x) \end{cases} \tag{2.13}$$

for $X = X(p,x)$, $P = P(p,x)$ as smooth functions of p, x, then

$$\mathbf{X}(t) := X(\mathbf{p}(t),\mathbf{x}(t)),\ \mathbf{P}(t) := P(\mathbf{p}(t),\mathbf{x}(t))$$

solve the dynamics

$$\begin{cases} \dot{\mathbf{X}} = D\bar{H}(P) \\ \dot{\mathbf{P}} = 0. \end{cases} \tag{2.14}$$

In other words, u is a generating function for a canonical transformation from the variables (p, x) to (P, X), with respect to which the new Hamiltonian dynamics (2.14) have the trivial solution

$$\mathbf{X}(t) = X_0 + tD\bar{H}(\mathbf{P}), \ \ \mathbf{P}(t) \equiv P.$$

It is in general impossible to carry out this process, but we may ask to what extent our PDE/variational methods identify some sort of "approximately integrable" dynamics.

2.5 Approximate dynamics, approximate canonical change of variable. We introduce the solution $\mathbf{x}^k = \mathbf{x}^k(P, t)$ of the ODE flow on $\mathbb{R}^n$

$$\dot{\mathbf{x}}^k = D_pH(D_xu^k(P, \mathbf{x}^k), \mathbf{x}^k), \quad \mathbf{x}^k(0) = x. \tag{2.15}$$

We also put

$$\mathbf{p}^k := D_xu^k(P, \mathbf{x}^k); \tag{2.16}$$

so that

$$\dot{\mathbf{x}}^k = D_pH(\mathbf{p}^k, \mathbf{x}^k). \tag{2.17}$$

Finally, define $\mathbf{X}^k = \mathbf{X}^k(P, t)$ by

$$\mathbf{X}^k := D_Pu^k(P, \mathbf{x}^k). \tag{2.18}$$

We ask to what extent $\mathbf{p}^k$ solves the second equation in (1.3) and $\mathbf{X}^k$ solves the first equation in (2.14). The keys to understanding these questions are two identities from [E1]. The first is formula (3.3) from [E1]:

$$\int_{\mathbb{T}^n} (H_{p_ip_j}u^k_{x_ix_l}u^k_{x_jx_l} + k|DH|^2)\sigma^k \, dx = -\int_{\mathbb{T}^n} (2H_{p_ix_i}u^k_{x_ix_j} + H_{x_ix_i})\sigma^k \, dx \tag{2.19}$$

for $DH := D_xH + D_pHD^2_xu^k$ and H evaluated at (Du^k, x). The second identity we will need is equation (4.3) in [E1]:

$$\begin{aligned} D^2\bar{H}^k(P) \ &= \ k\int_{\mathbb{T}^n} (D_pH(Du^k, x)D^2_{xP}u^k - D\bar{H}^k(P)) \\ &\quad \otimes(D_pH(Du^kx)D^2_{xP}u^k - D\bar{H}^k(P))\sigma^k \, dx \\ &\quad + \int_{\mathbb{T}^n} D^2_pH(Du^k, x)D^2_{xP}u^k \otimes D^2_{xP}u^k\sigma^k \, dx. \end{aligned} \tag{2.20}$$

Theorem 2.1 (i) *For each $R > 0$, there exists a constant C_R such that*

$$\int_{\mathbb{T}^n} |\dot{\mathbf{p}}^k + D_x H(\mathbf{p}^k, \mathbf{x}^k)|^2 \sigma^k dx \le \frac{C_R}{k} \tag{2.21}$$

for all $t \in \mathbb{R}$ and $|P| \le R$.

(ii) *There exists a constant C such that*

$$\int_{B(0,R)} \int_{\mathbb{T}^n} |\dot{\mathbf{X}}^k - D\bar{H}^k(P)|^2 \sigma^k \, dx dP \le \frac{C_R}{k} \tag{2.22}$$

for all $t \in \mathbb{R}$ and $R > 0$. Therefore

$$\int_{B(0,R)} \int_{\mathbb{T}^n} \max_{|t| \le T} |\mathbf{X}^k - X_0^k - tD\bar{H}^k(P)|^2 \sigma^k \, dx dP \le \frac{C_R T}{k} \tag{2.23}$$

for $X_0^k = \mathbf{X}^k(0)$ and for each time $T > 0$.

Interpretations. (i) According to (2.17), the functions $\mathbf{x}^k$ exactly solve the first of Hamilton's equations (1.3). We understand (2.21) as providing a quantitative estimate showing that the functions $\mathbf{p}^k$ are approximate solutions of the second of Hamilton's equations, at least for initial data where σ^k has positive mass in the limit $k \to \infty$.

(ii) Obviously $\mathbf{P} \equiv P$ exactly solves the second of Hamilton's equations (2.14), transformed into the new variables (X, P). We interpret (2.22) as asserting that the functions $\mathbf{X}^k$ are approximate solutions of the first of the equations (2.14), corresponding to initial data where σ^k has positive mass in the limit $k \to \infty$. In this weak sense, the smooth function u^k acts like an approximate generating function, selecting out "approximately integrable" dynamics. It would be extremely interesting to make this assertion more precise.

Proof. 1. We compute

$$\dot{\mathbf{p}}^k + D_x H = D_x^2 u^k D_p H + D_x H = DH.$$

Thus

$$\int_{\mathbb{T}^n} |\dot{\mathbf{p}}^k + D_x H|^2 \sigma^k \, dx = \int_{\mathbb{T}^n} |DH|^2 \sigma^k \, dx,$$

the integrand evaluated at $(\mathbf{p}^k, \mathbf{x}^k) = (Du^k(\mathbf{x}^k), \mathbf{x}^k)$. But since $\operatorname{div}(\sigma^k D_p H) = 0$, the measure $\sigma^k dx$ is flow invariant. So

$$\int_{\mathbb{T}^n} |\dot{\mathbf{p}}^k + D_x H|^2 \sigma^k \, dx = \int_{\mathbb{T}^n} |DH|^2 \sigma^k \, dx, \tag{2.24}$$

H evaluated at $(Du^k(x), x)$.

Now according to (2.19), we have the estimate

$$\int_{\mathbb{T}^n} (|D_x^2 u^k|^2 + k|DH|^2)\sigma^k \, dx \leq C \int_{\mathbb{T}^n} (|D_{xp}^2 H|^2 + |D_x^2 H|)\sigma^k \, dx \leq C_R, \tag{2.25}$$

provided $|P| \leq R$. This and (2.24) imply (2.21).

2. We have

$$\dot{\mathbf{X}}^k = D_{xP}^2 u^k \dot{\mathbf{x}}^k = D_{xP}^2 u^k D_p H;$$

and consequently for fixed P,

$$\int_{\mathbb{T}^n} |\dot{\mathbf{X}}^k - D\bar{H}^k|^2 \sigma^k \, dx = \int_{\mathbb{T}^n} |D_{xP}^2 u^k D_p H - D\bar{H}^k|^2 \sigma^k \, dx.$$

In the integrand u^k is evaluated at $x = \mathbf{x}^k$ and H is evaluated at $(\mathbf{p}^k, \mathbf{x}^k) = (Du^k(\mathbf{x}^k), \mathbf{x}^k)$. These expressions depend upon $t \in \mathbb{R}$ and the initial point $\mathbf{x}^k(0) = x$ for the flow (1.3). But the flow invariance of $\sigma^k dx$ implies

$$\int_{\mathbb{T}^n} |\dot{\mathbf{X}}^k - D\bar{H}^k(P)|^2 \sigma^k dx = \int_{\mathbb{T}^n} |D_{xP}^2 u^k D_p H - D\bar{H}^k|^2 \sigma^k \, dx, \tag{2.26}$$

where now u^k and H are evaluated at x and $(Du^k(x), x)$. In view therefore of (2.20), we have the inequality

$$\int_{\mathbb{T}^n} |\dot{\mathbf{X}}^k - D\bar{H}^k(P)|^2 \sigma^k \, dx \leq \frac{1}{k} tr(D^2 \bar{H}^k(P));$$

and then

$$\int_{B(0,R)} \int_{\mathbb{T}^n} |\dot{\mathbf{X}}^k - D\bar{H}^k(P)|^2 \sigma^k dx dP \leq \frac{1}{Rk} \int_{\partial B(0,R)} D\bar{H}^k(P) \cdot P \, d\mathcal{H}^{n-1}. \tag{2.27}$$

Finally observe from (2.20) that $P \mapsto \bar{H}^k(P)$ is convex; whence follows the estimate

$$\max_{B(0,R)} |D\bar{H}^k| \leq \frac{C}{R} \max_{B(0,2R)} |\bar{H}^k|. \tag{2.28}$$

Furthermore $0 \leq H \leq C(|P|^2 + 1)$ implies $0 \leq \bar{H}(P) \leq C(|P|^2 + 1)$. Since [E1, Theorem 4.1] implies $\bar{H}^k \leq \bar{H}$, we have

$$\bar{H}^k(P) \leq C(|P|^2 + 1). \tag{2.29}$$

Also

$$\bar{H}^k(P) \geq \frac{1}{k} \log(|\mathbb{T}^n|) = 0. \tag{2.30}$$

Hence (2.27)–(2.30) imply the stated estimate (2.22). □

3 Hamiltonian approximation by principal eigenvalues

3.1 A new approximation. This and the next section introduce alternative variational principles, based upon two regularizations using various forms of the Donsker–Varadhan [D-V] I-functional.

Let us first recall that $\frac{1}{2}\Delta$ is the infinitesimal generator of Brownian motion and that the corresponding Donsker–Varadhan I-functional for probability measures μ on $\mathbb{T}^n$ is

$$I[\mu] = -\inf_{\phi>0}\int_{\mathbb{T}^n}\frac{\Delta\phi}{2\phi}\,d\mu = \begin{cases}\frac{1}{2}\int_{\mathbb{T}^n}|D\psi|^2\,dx & \text{if } d\mu=\psi^2dx\\ +\infty & \text{otherwise.}\end{cases} \tag{3.1}$$

We introduce next for $\varepsilon>0$ and smooth functions v the functional

$$J_\varepsilon[v] := \sup_{\mu}\left\{\int_{\mathbb{T}^n} H(P+Dv,x)\,d\mu - \varepsilon I[\mu]\right\}, \tag{3.2}$$

which should be compared with the entropy regularization (2.2) that leads to (2.1). In view of (3.1), for each function v we have

$$J_\varepsilon[v] = -\min_{\psi}\left\{\int_{\mathbb{T}^n}\frac{\varepsilon}{2}|D\psi|^2 - H(P+Dv,x)\psi^2\,dx \mid \int_{\mathbb{T}^n}\psi^2\,dx=1\right\}. \tag{3.3}$$

We select v^ε to minimize $J_\varepsilon[\cdot]$ among functions with mean zero over $\mathbb{T}^n$. That is, we take $v=v^\varepsilon$ so that the corresponding principal eigenvalue $\lambda=J_\varepsilon[v]$ of the problem

$$\frac{\varepsilon}{2}\Delta w + H(P+Dv,x)w = \lambda w$$

is minimized. Let λ_ε denote this minimal value of the principal eigenvalue, and w^ε be the corresponding principal eigenfunction. Then

$$\frac{\varepsilon}{2}\Delta w^\varepsilon + H(P+Dv^\varepsilon,x)w^\varepsilon = \lambda_\varepsilon w^\varepsilon \tag{3.4}$$

on the torus $\mathbb{T}^n$, normalized so that

$$w^\varepsilon>0,\quad \int_{\mathbb{T}^n}(w^\varepsilon)^2\,dx=1. \tag{3.5}$$

For later reference we define

$$\sigma^\varepsilon := (w^\varepsilon)^2,\quad u^\varepsilon := P\cdot x + v^\varepsilon. \tag{3.6}$$

We will show that versions of the basic PDE (1.1), (1.2) of weak KAM theory are hidden within this new minimization problem.

We will hereafter simply assume that the minimizer v^ε exists and is smooth, and thus $u^\varepsilon = P \cdot x + v^\varepsilon$ and the corresponding principal eigenfunction w^ε are smooth. It seems possible to prove for each $\varepsilon > 0$ the regularity of u^ε and w^ε using the PDE (3.4) and (3.7) (derived below), but a developing a full proof would be a distraction from the main issue, the derivation of weak KAM theory in the limit $\varepsilon \to 0$.

3.2 First variation. The first variation of our problem produces the Euler-Lagrange equation:

Theorem 3.1 *The density σ^ε solves the PDE*

$$\operatorname{div}(D_p H(Du^\varepsilon, x)\sigma^\varepsilon) = 0. \tag{3.7}$$

Proof. To simplify notation, we drop the superscripts ε; so that (3.4) reads

$$\frac{\varepsilon}{2}\Delta w + H(Du, x)w = \lambda w, \tag{3.8}$$

where w and λ depend upon u. Let $\{u(\tau) \mid |\tau| \leq 1\}$ be a smooth curve of functions, with $u(0) = u$. Let $\lambda(\tau)$ denote the principal eigenvalue corresponding to the potential $H(Du(\tau), x)$:

$$\frac{\varepsilon}{2}\Delta w(\tau) + H(Du(\tau), x)w(\tau) = \lambda(\tau)w(\tau).$$

Since the principal eigenvalue is simple, we can take $\lambda(\cdot)$ and $w(\cdot)$ to be smooth functions of τ.

Differentiating with respect to τ and then putting $\tau = 0$, we find

$$\frac{\varepsilon}{2}\Delta w'(0) + H(Du, x)w'(0) - \lambda(0)w'(0) = D_p H(Du, x) \cdot Du'(0)w(0) + \lambda'(0)w(0).$$

Multiply by $w = w(0)$ and integrate by parts, recalling (3.5) and (3.6) to discover the identity

$$\lambda'(0) = -\int_{\mathbb{T}^n} D_p H(Du, x) \cdot Du'(0)\sigma \, dx.$$

Since $u = u(0)$ minimizes the principal eigenvalue λ, we have $\lambda'(0) = 0$. Consequently,

$$\int_{\mathbb{T}^n} D_p H(Du, x) \cdot Du'(0)\sigma \, dx = 0$$

for all variations $u'(0)$. This implies the weak formulation of (3.7). $\square$

3.3 Second variation. A second variation provides us with bounds on the second derivatives of u^ε:

Theorem 3.2 (i) *We have the estimate*

$$\int_{\mathbb{T}^n} |Du^\varepsilon|^2 \sigma^\varepsilon \, dx \leq C, \tag{3.9}$$

the constant C *independent of* ε.

(ii)*Furthermore,*

$$\int_{\mathbb{T}^n} |D^2 u^\varepsilon|^2 \sigma^\varepsilon \, dx \leq C. \tag{3.10}$$

Proof. 1. Again we omit the superscript ε. Multiply the PDE (3.1) by the periodic function $v = v^\varepsilon$ and integrate:

$$\int_{\mathbb{T}^n} D_p H(Du, x) \cdot Dv \sigma \, dx = 0.$$

Estimate (3.9) follows, since $Du = P + Dv$ and since the uniform convexity of H in the variable p implies $|p|^2 \leq C(D_p H(p,x) \cdot p + 1)$ for some positive constant C.

2. Differentiate (3.8) once, and then twice, with respect to x_k:

$$\frac{\varepsilon}{2} \Delta w_{x_k} + H w_{x_k} + (H)_{x_k} w = \lambda w_{x_k}, \tag{3.11}$$

$$\frac{\varepsilon}{2} \Delta w_{x_k x_k} + H w_{x_k x_k} + 2(H)_{x_k} w_{x_k} + (H)_{x_k x_k} w = \lambda w_{x_k x_k}. \tag{3.12}$$

Here we use the notation $(H)_{x_k} = (H(Du, x))_{x_k}$. Multiplying (3.12) by w, integrating by parts and using (3.11), we deduce

$$\int_{\mathbb{T}^n} 2(H)_{x_k} w_{x_k} w + (H)_{x_k x_k} w^2 \, dx = 0.$$

Consequently, using (3.11) we see that

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{T}^n} (H)_{x_k x_k} w^2 \, dx &= - \int_{\mathbb{T}^n} ((H)_{x_k} w) w_{x_k} \, dx \\
&= \int_{\mathbb{T}^n} \left(\frac{\varepsilon}{2} \Delta w_{x_k} + H w_{x_k} - \lambda w_{x_k} \right) w_{x_k} \, dx \\
&= - \int_{\mathbb{T}^n} \frac{\varepsilon}{2} |D w_{x_k}|^2 dx + (\lambda - H) w_{x_k}^2 \, dx.
\end{aligned} \tag{3.13}$$

Now

$$\lambda = J_\varepsilon[v] = - \min_\psi \left\{ \frac{\int_{\mathbb{T}^n} \frac{\varepsilon}{2} |D\psi|^2 - H \psi^2 \, dx}{\int_{\mathbb{T}^n} \psi^2 \, dx} \right\}$$

and therefore

$$\int_{\mathbb{T}^n} \frac{\varepsilon}{2}|D\psi|^2 + (\lambda - H)\psi^2 \, dx \geq 0 \tag{3.14}$$

for all periodic functions ψ. In particular,

$$\int_{\mathbb{T}^n} \frac{\varepsilon}{2}|Dw_{x_k}|^2 + (\lambda - H)w_{x_k}^2 \, dx \geq 0$$

for $k = 1, \ldots, n$; and so (3.13) implies

$$\int_{\mathbb{T}^n} (H)_{x_k x_k} \sigma \, dx \leq 0.$$

But

$$(H)_{x_k x_k} = H_{p_i} u_{x_i x_k x_k} + H_{p_i p_j} u_{x_i x_k} u_{x_j x_k} + 2H_{p_i x_k} u_{x_i x_k} + H_{x_l x_k}.$$

We substitute above, and note that Theorem 3.1 implies

$$\int_{\mathbb{T}^n} H_{p_i} u_{x_i x_k x_k} \sigma \, dx = 0.$$

Estimate (3.10) then follows from (1.4), the strict convexity of H in the variable p and (3.9). □

3.4 Differentiations in the variable P. Next define

$$\bar{H}^\varepsilon(P) := \lambda_\varepsilon = J_\varepsilon[v^\varepsilon]. \tag{3.15}$$

We will later show that the function $\bar{H}^\varepsilon$ is an approximation to the effective Hamiltonian $\bar{H}$.

Theorem 3.3 (i) *We have*

$$D\bar{H}^\varepsilon(P) = \int_{\mathbb{T}^n} D_p H(Du^\varepsilon, x)\sigma^\varepsilon \, dx. \tag{3.16}$$

(ii) *Furthermore,*

$$D^2\bar{H}^\varepsilon(P) = \int_{\mathbb{T}^n} D_p^2 H D_{xP}u^\varepsilon \otimes D_{xP}u^\varepsilon \sigma^\varepsilon + \varepsilon D_{xP}w^\varepsilon \otimes D_{xP}w^\varepsilon + 2(\lambda - H)D_P w^\varepsilon \otimes D_P w^\varepsilon \, dx. \tag{3.17}$$

It follows that the mapping $P \mapsto \bar{H}^\varepsilon(P)$ is convex, since for all $\xi \in \mathbb{R}^n$

$$\int_{\mathbb{T}^n} \varepsilon|D_P w \cdot \xi|^2 + 2(\lambda - H)|D_{xP} w \cdot \xi|^2 \, dx \geq 0$$

according to (3.14).

Proof. 1. As usual, we drop the superscripts ε. Differentiate (3.8) with respect to P_k, to find

$$\frac{\varepsilon}{2}\Delta w_{P_k} + H w_{P_k} + (H)_{P_k} w = \lambda w_{P_k} + \lambda_{P_k} w. \tag{3.18}$$

Multiply by w, integrate by parts, and recall that we are now writing $\bar{H}^\varepsilon = \lambda$:

$$\bar{H}^\varepsilon_{P_k} = \int_{\mathbb{T}^n} H_{p_i} u_{x_i P_k} \sigma dx = \int_{\mathbb{T}^n} H_{p_k}(Du, x)\sigma\, dx,$$

the last equality holding in view of (3.7), since $u = P \cdot x + v$ and v is periodic in x.

2. Next, differentiate (3.18) with respect to P_l:

$$\begin{aligned}\frac{\varepsilon}{2}\Delta w_{P_k P_l} + H w_{P_k P_l} + (H)_{P_k} w_{P_l} + (H)_{P_l} w_{P_k} + (H)_{P_k P_l} w \\ = \lambda w_{P_k P_l} + \lambda_{P_k} w_{P_l} + \lambda_{P_l} w_{P_k} + \lambda_{P_k P_l} w.\end{aligned}$$

We discover upon multiplying by w and integrating by parts that

$$\bar{H}^\varepsilon_{P_k P_l} = \int_{\mathbb{T}^n} ((H)_{P_k} w_{P_l} + (H)_{P_l} w_{P_k}) w + (H)_{P_k P_l} w^2\, dx,$$

since

$$\int_{\mathbb{T}^n} w_{P_k} w dx = \frac{1}{2}\frac{\partial}{\partial P_k}\int_{\mathbb{T}^n} w^2 v dx = 0.$$

Recalling (3.18), we further calculate that

$$\begin{aligned}\bar{H}^\varepsilon_{P_k P_l} &= \int_{\mathbb{T}^n} ((H)_{P_k} w) w_{P_l} + ((H)_{P_l} w) w_{P_k} + (H)_{P_k P_l} w^2\, dx \\ &= \int_{\mathbb{T}^n} \left((\lambda - H) w_{P_k} - \frac{\varepsilon}{2}\Delta w_{P_k}\right) w_{P_l} + \\ &\qquad \left((\lambda - H) w_{P_l} - \frac{\varepsilon}{2}\Delta w_{P_l}\right) w_{P_k} + (H)_{P_k P_l} w^2\, dx \\ &= \int_{\mathbb{T}^n} \varepsilon D_x w_{P_k} \cdot D_x w_{P_l} + 2(\lambda - H) w_{P_k} w_{P_l} + H_{p_i p_j} u_{x_i P_k} u_{x_j P_l} w^2\, dx.\end{aligned}$$

□

3.5 Limits as $\epsilon \to 0$. We next show that in the limit $\varepsilon \to 0$ the basic PDE of weak KAM theory appear.

Firstly, let us introduce for this section the temporary notation that the Lipschitz continuous function $\hat{u}$ and the measure $\hat{\sigma}$ are weak solutions of (1.1) and (1.2):

$$H(D\hat{u}, x) = \bar{H}(P), \tag{3.19}$$

in the viscosity sense and also $\hat{\sigma}$ almost everywhere, where $\hat{u} = P \cdot x + \hat{v}$ for periodic $\hat{v}$; and

$$\operatorname{div}(D_p H(D\hat{u}, x)\hat{\sigma}) = 0. \tag{3.20}$$

At this point we want to introduce as in Chapter 2 a probability measure ν satisfying (2.8). There is however a problem since for the current approximation based upon (3.2), unlike the alternate approximations in Section 2 above and Section 4 below, we do not have uniform sup-norm bounds on $|Du^\varepsilon|$. However according to (3.9), we do have uniform L^2 bounds on $|Du^\varepsilon|$ if we integrate against the measure σ^e. The next lemma shows that these are good enough if we take test functions Φ in (2.8) that grow at most quadratically in p.

Lemma 3.1 (i) *We have the bound*

$$\int_{\mathbb{T}^n} |Du^\varepsilon|^2 \psi^2 \, dx \le C \int_{\mathbb{T}^n} \varepsilon |D\psi|^2 + \psi^2 \, dx + C \tag{3.21}$$

for each smooth function ψ.

(ii) *In addition,*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{T}^n} |D\hat{u} - Du^\varepsilon|^2 \sigma^\varepsilon \, dx = 0. \tag{3.22}$$

Proof. 1. The PDE (3.4) implies

$$\frac{\varepsilon}{2w^\varepsilon} \Delta w^\varepsilon + H(Du^\varepsilon, x) = \bar{H}^\varepsilon(P).$$

Multiply by ψ^2 and integrate:

$$\frac{\varepsilon}{2} \int_{\mathbb{T}^n} \frac{|Dw^\varepsilon|^2}{(w^\varepsilon)^2} \psi^2 \, dx + \int_{\mathbb{T}^n} H(Du^\varepsilon, x) \psi^2 \, dx = \bar{H}^\varepsilon(P) \int_{\mathbb{T}^n} \psi^2 \, dx + \varepsilon \int_{\mathbb{T}^n} \frac{\psi}{w^\varepsilon} Dw^\varepsilon \cdot D\psi \, dx.$$

Since $|p|^2 \le H(p, x) + C$, this implies the estimate (3.21).

2. The uniform convexity hypothesis (1.4) implies

$$H(Du^\varepsilon, x) + D_p H(Du^\varepsilon, x) \cdot (D\hat{u} - Du^\varepsilon) + \frac{\gamma}{2} |D\hat{u} - Du^\varepsilon|^2 \le H(D\hat{u}, x) = \bar{H}(P).$$

Consequently,

$$\frac{\gamma}{2}\int_{\mathbb{T}^n}|D\hat{u}-Du^\varepsilon|^2\sigma^\varepsilon\,dx+\int_{\mathbb{T}^n}H(Du^\varepsilon,x)\sigma^\varepsilon\,dx\leq\bar{H}(P).$$

Now multiply (3.4) by w^ε and integrate:

$$\int_{\mathbb{T}^n}H(Du^\varepsilon,x)\sigma^\varepsilon\,dx=\bar{H}^\varepsilon(P)+\frac{\varepsilon}{2}\int_{\mathbb{T}^n}|Dw^\varepsilon|^2\,dx.$$

Therefore

$$\int_{\mathbb{T}^n}|D\hat{u}-Du^\varepsilon|^2\sigma^\varepsilon\,dx\leq C|\bar{H}(P)-\bar{H}^\varepsilon(P)|\to 0,$$

since we will see in the proof of Theorem 3.4 below that $\bar{H}^\varepsilon(P)\to\bar{H}(P)$. □

In view of this result, we may assume upon passing if necessary to a subsequence that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{T}^n}\Phi(Du^\varepsilon,x)\sigma^\varepsilon\,dx=\int_{\mathbb{R}^n}\int_{\mathbb{T}^n}\Phi(p,x)\,d\nu \tag{3.23}$$

for all continuous functions $\Phi=\Phi(p,x)$ satisfying the quadratic growth bound $|\Phi(p,x)|\leq C(|p|^2+1)$. We define also the probability measure μ to satisfy

$$\int_{\mathbb{R}^n}\int_{\mathbb{T}^n}\Psi(v,x)\,d\mu=\int_{\mathbb{R}^n}\int_{\mathbb{T}^n}\Psi(D_pH(p,x),x)\,d\nu \tag{3.24}$$

for all continuous $\Psi=\Psi(v,x)$ growing at most quadratically in v.

Next is our main assertion, that the measure μ defined by (3.23) and (3.24) is a minimizing measure.

Theorem 3.4 (i) *We have*

$$\lim_{\varepsilon\to 0}\bar{H}^\varepsilon(P)=\bar{H}(P). \tag{3.25}$$

(ii) *The measure μ satisfies*

$$\int_{\mathbb{R}^n}\int_{\mathbb{T}^n}v\cdot D\phi\,d\mu=0 \tag{3.26}$$

for all smooth, periodic functions $\phi=\phi(x)$; and

$$\int_{\mathbb{R}^n}\int_{\mathbb{T}^n}v\,d\mu=:V\in\partial\bar{H}(P). \tag{3.27}$$

(iii) *The measure μ minimizes Mather's action functional* (2.11) *among all other probability measures satisfying* (3.26) *and* (3.27).

General weak KAM theory (as recounted for instance in [E3]) then implies that the measure ν has the form $\nu = \delta_{\{p=D\hat{u}\}}\sigma$. This by the way already follows from our estimate (3.22).

Proof. 1. We have

$$\bar{H}^\varepsilon(P) = \inf_v \sup_\psi \left\{ \int_{\mathbb{T}^n} H(P+Dv,x)\psi^2 - \frac{\varepsilon}{2}|D\psi|^2\,dx \mid \int_{\mathbb{T}^n} \psi^2\,dx = 1 \right\};$$

and so our taking $v = \hat{v}$ shows

$$\bar{H}^\varepsilon(P) \le \bar{H}(P).$$

A bound from below is harder. For this, note that

$$\begin{aligned}\bar{H}^\varepsilon(P) &= \sup_\psi \left\{ \int_{\mathbb{T}^n} H(P+Dv^\varepsilon,x)\psi^2 - \frac{\varepsilon}{2}|D\psi|^2\,dx \mid \int_{\mathbb{T}^n} \psi^2\,dx = 1 \right\} \\ &\ge \int_{\mathbb{T}^n} H(P+Dv^\varepsilon,x)\psi^2 dx - \frac{\varepsilon}{2}|D\psi|^2\,dx,\end{aligned}$$

for each smooth function ψ with L^2 norm equaling one. Now in view of estimate (3.21), we may assume

$$Du^\varepsilon = P + Dv^\varepsilon \rightharpoonup Du = P + Dv \qquad \text{weakly in } L^2$$

for some periodic, Lipschitz continuous function v. Therefore lower semicontinuity of the integral implies

$$\liminf_{\varepsilon\to 0} \bar{H}^\varepsilon(P) \ge \int_{\mathbb{T}^n} H(Du,x)\psi^2\,dx.$$

Recall next the smooth functions $u^k = P\cdot x + v^k$ and σ^k introduced in Section 2. Our taking $\psi^2 = \sigma^k$ shows that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \bar{H}^\varepsilon(P) &\ge \int_{\mathbb{T}^n} H(Du,x)\sigma^k\,dx \\ &\ge \int_{\mathbb{T}^n} H(Du^k,x)\sigma^k\,dx + \int_{\mathbb{T}^n} DH(Du^k,x)\cdot D(v-v^k)\sigma^k\,dx.\end{aligned}$$

Owing to (2.3), the last term equals zero. We now send $k\to\infty$ and recall (2.5), to deduce that

$$\liminf_{\varepsilon\to 0} \bar{H}^\varepsilon(P) \ge \bar{H}(P).$$

This proves (3.25).

2. According to Theorem 3.1,

$$\int_{\mathbb{T}^n} D_pH(Du^\varepsilon,x)\cdot D\phi\sigma^\varepsilon\,dx = 0,$$

for all periodic ϕ. Remember (3.23) and send $\varepsilon \to 0$:

$$\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} D_pH(p,x)\cdot D\phi\, d\nu = 0.$$

Changing to the (v,x) variables and recalling the definition of the measure μ gives us (3.26).

Next, recall (3.16):

$$D\bar{H}^\varepsilon(P) = \int_{\mathbb{T}^n} D_pH(Du^\varepsilon, x)\sigma^\varepsilon\, dx.$$

The limit of the right hand side as $\varepsilon \to 0$ is

$$\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} D_pH(p,x)\, d\nu = \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} v\, d\mu =: V.$$

Since the functions $\bar{H}^\varepsilon$ are convex and converge pointwise to $\bar{H}$, it follows that $V \in \partial\bar{H}(P)$.

3. We now assert

$$\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} L(v,x)\, d\mu + \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} H(p,x)\, d\nu = P\cdot V. \tag{3.28}$$

To see this, notice that the term on the left equals

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} L(D_pH,x) + H(p,x)\, d\nu &= \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} D_pH\cdot p\, d\nu \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{T}^n} D_pH(Du^\varepsilon,x)\cdot Du^\varepsilon\sigma^\varepsilon\, dx \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{T}^n} D_pH(Du^\varepsilon,x)\cdot(P+Dv^\varepsilon)\sigma^\varepsilon\, dx \\ &= P\cdot V.\end{aligned}$$

Next, multiply (3.4) by w^ε and integrate by parts:

$$\bar{H}^\varepsilon(P) = \lambda^\varepsilon = \int_{\mathbb{T}^n} H(Du^\varepsilon,x)\sigma^\varepsilon\, dx - \frac{\varepsilon}{2}\int_{\mathbb{T}^n} |Dw^\varepsilon|^2\, dx.$$

This identity and (3.25) imply

$$\bar{H}(P) \le \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} H\, d\nu.$$

Therefore we can invoke (3.28) to calculate

$$\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} L\, d\mu = P\cdot V - \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} H\, d\nu \le P\cdot V - \bar{H}(P) = \bar{L}(V).$$

where we recall that the effective Lagrangian $\bar{L}$ is the dual of $\bar{H}$. The last equality holds since $V \in \partial\bar{H}(P)$. But $\bar{L}(V)$ is the minimum value of the action (2.11) among all probability measures satisfying (3.26), (3.27), and so it follows that μ is in fact a minimizer. $\square$

4 Lagrangian approximation by principal eigenvalues

4.1 A different approximation. Inspired in part by Benamou and Brenier [B-B], we present now a Lagrangian variant of the foregoing approximation. The main new technical feature is that the symmetric eigenvalue problem (3.4) is replaced by two dual eigenvalue problems (4.4) for a nonsymmetric operator. These computations are motivated by somewhat related dual eigenfunction calculations in [E2]; and this section represents a partial solution to the problem of generalizing the approach of that paper to Hamiltonians more general than $\frac{1}{2}|p|^2 + W(x)$.

Fix a vector field $\mathbf{v} = \mathbf{v}(x)$ on $\mathbb{T}^n$, and introduce the generator of corresponding flow, regularized by an ε-dependent viscosity term:

$$A^\varepsilon_{\mathbf{v}}\phi := \mathbf{v} \cdot D\phi + \varepsilon\Delta\phi. \tag{4.1}$$

The corresponding Donsker–Varadhan I-functional is

$$I^\varepsilon_{\mathbf{v}}[\mu] := -\inf_{\phi>0} \int_{\mathbb{T}^n} \frac{A^\varepsilon_{\mathbf{v}}\phi}{\phi}\, d\mu. \tag{4.2}$$

We introduce next for $\varepsilon > 0$ and $P \in \mathbb{R}^n$ the expression

$$K_\varepsilon[\mathbf{v}] := -\min_{\mu} \left\{ \int_{\mathbb{T}^n} L(\mathbf{v}, x) - P \cdot \mathbf{v}\, d\mu + I^\varepsilon_{\mathbf{v}}[\mu] \right\}, \tag{4.3}$$

the minimum taken over probability measures μ on $\mathbb{T}^n$. As we will see, the effect of the term $I^\varepsilon_{\mathbf{v}}[\cdot]$ in the limit $\varepsilon \to 0$ will be to enforce the flow invariance requirement (2.12). The Donsker-Varadhan formula asserts that $K_\varepsilon[\mathbf{v}]$ equals the principle eigenvalue of the operator $\varepsilon\Delta + \mathbf{v} \cdot D - (L(\mathbf{v}, x) - P \cdot \mathbf{v})$ on $\mathbb{T}^n$.

We select $\mathbf{v}_\varepsilon$ to minimize $K_\varepsilon[\cdot]$ among vector fields over $\mathbb{T}^n$. That is, we take $\mathbf{v}_\varepsilon$ to minimize the corresponding principal eigenvalue λ^ε of the dual problems

$$\begin{cases} \varepsilon\Delta w^\varepsilon + \mathbf{v}_\varepsilon \cdot Dw^\varepsilon - (L(\mathbf{v}_\varepsilon, x) - P \cdot \mathbf{v}_\varepsilon)w^\varepsilon = \lambda^\varepsilon w^\varepsilon \\ \varepsilon\Delta w^{\varepsilon*} - \operatorname{div}(\mathbf{v}_\varepsilon w^{\varepsilon*}) - (L(\mathbf{v}_\varepsilon, x) - P \cdot \mathbf{v}_\varepsilon)w^{\varepsilon*} = \lambda^\varepsilon w^{\varepsilon*}. \end{cases} \tag{4.4}$$

The dual eigenfunctions w^ε and $w^{\varepsilon*}$ are positive and are normalized so that

$$\int_{\mathbb{T}^n} w^\varepsilon w^{\varepsilon*}\, dx = 1. \tag{4.5}$$

For later reference we define

$$\sigma^\varepsilon := w^\varepsilon w^{\varepsilon*}, \quad v^\varepsilon := \log w^\varepsilon, \quad u^\varepsilon := P \cdot x + v^\varepsilon. \tag{4.6}$$

As in Section 3 we will just assume that the minimizer $\mathbf{v}_\varepsilon$ exists and is smooth, although Theorem 4.1 will show that we can in fact compute $\mathbf{v}_\varepsilon$ in terms of u^ε, which turns out to be the smooth solution of the PDE (4.9).

4.2 First variation. As usual, the first variation provides useful information:

Theorem 4.1 (i) *We have*

$$D_v L(\mathbf{v}_\varepsilon, x) = P + \frac{Dw^\varepsilon}{w^\varepsilon} = Du^\varepsilon, \tag{4.7}$$

and consequently

$$D_p H(Du^\varepsilon, x) = \mathbf{v}_\varepsilon. \tag{4.8}$$

(ii) *Furthermore, u^ε solves the PDE*

$$\varepsilon \Delta u^\varepsilon + \varepsilon |Du^\varepsilon - P|^2 + H(Du^\varepsilon, x) = \lambda^\varepsilon. \tag{4.9}$$

Proof. 1. To simplify notation, we drop the sub- and superscripts ε. Then the first equation in (4.4) says

$$\varepsilon \Delta w + \mathbf{v} \cdot Dw - (L(\mathbf{v}, x) - P \cdot \mathbf{v}) w = \lambda w, \tag{4.10}$$

where w and λ depend upon $\mathbf{v}$. Let $\{\mathbf{v}(\tau) \mid |\tau| \leq 1\}$ be a smooth curve of vector fields, with $\mathbf{v}(0) = \mathbf{v}$. Let $\lambda(\tau)$ denote the corresponding principal eigenvalue. Then

$$\varepsilon \Delta w(\tau) + \mathbf{v}(\tau) \cdot Dw(\tau) - (L(\mathbf{v}(\tau), x) - P \cdot \mathbf{v}(\tau)) w(\tau) = \lambda(\tau) w(\tau).$$

Differentiating with respect to τ and then setting $\tau = 0$, we discover

$$\begin{aligned} \varepsilon \Delta w'(0) + \mathbf{v}'(0) \cdot Dw + \mathbf{v} \cdot Dw'(0) - (D_v L(\mathbf{v}, x) - P) \cdot \mathbf{v}'(0) w \\ - (L(\mathbf{v}, x) - P \cdot \mathbf{v}) w'(0) = \lambda w'(0) + \lambda'(0) w. \end{aligned} \tag{4.11}$$

Multiply by the dual eigenfunction $w^* = w^{\varepsilon *}$ and integrate by parts, using the second equation in (4.4) to remove the expressions involving $w'(0)$ and deduce

$$\lambda'(0) = \int_{\mathbb{T}^n} \mathbf{v}'(0) \cdot Dw w^* - (D_v L(\mathbf{v}, x) - P) \cdot \mathbf{v}'(0) w w^* \, dx.$$

We have $\lambda'(0) = 0$, since $\mathbf{v} = \mathbf{v}(0)$ minimizes the principal eigenvalue λ. Furthermore the variation $\mathbf{v}'(0)$ is arbitrary, and thus

$$(Dw - (D_v L(\mathbf{v}, x) - P) w) w^* = 0.$$

Since $w^* > 0$, assertion (4.7) follows.

2. In view of (4.7), $H(Du^\varepsilon, x) = \mathbf{v}_\varepsilon \cdot Du^\varepsilon - L(\mathbf{v}_\varepsilon, x)$; and so the first equation in (4.4) becomes

$$\varepsilon \Delta w^\varepsilon + H(Du^\varepsilon, x) w^\varepsilon = \lambda^\varepsilon w^\varepsilon. \tag{4.12}$$

Since $v^\varepsilon = \log w^\varepsilon$, we can rewrite this PDE into the form (4.9). □

Remarks. (i) Observe that the PDE (4.12) satisfied by $u^\varepsilon, w^\varepsilon$ agrees (up to a factor $\frac{1}{2}$) with the PDE (3.4) satisfied by the corresponding functions $u^\varepsilon, w^\varepsilon$ in Section 3. However our current function u^ε (defined by (4.6)) does not seem to correspond to a solution v^ε the minimization problem for the functional $J_\varepsilon[\cdot]$ discussed in Section 3.

(ii) We note also that in fact $w'(0) \equiv 0$ in the foregoing calculation; that is, an $O(\tau)$ variation of the minimizer $\mathbf{v}_\varepsilon$ creates an $o(\tau)$ variation in the principal eigenfunction w^ε.

To see this, observe that our plugging (4.7) into (4.11) gives

$$\varepsilon \Delta w'(0) + \mathbf{v} \cdot Dw'(0) - (L(\mathbf{v}, x) - P \cdot \mathbf{v}) w'(0) = \lambda w'(0).$$

Consequently, since the eigenspace for the principal eigenvalue is one dimensional, we have $w'(0) = \kappa w = \kappa w^\varepsilon$ for some constant κ. But the normalization $\int w(\tau)^2 \, dx = 1$ implies $\int w(0) w'(0) \, dx = 0$ and therefore $\kappa = 0$. □

We derive next a form of the Euler-Lagrange equation:

Theorem 4.2 *The density σ^ε solves the PDE*

$$-\varepsilon \Delta \sigma^\varepsilon + 2\varepsilon \operatorname{div}(\sigma^\varepsilon D v^\varepsilon) + \operatorname{div}(D_p H(Du^\varepsilon, x) \sigma^\varepsilon) = 0. \tag{4.13}$$

Proof. We again drop the superscripts ε, so that the dual eigenfunction equations (4.4) read

$$\begin{cases} \varepsilon \Delta w + \mathbf{v} \cdot Dw - (L(\mathbf{v}, x) - P \cdot \mathbf{v}) w = \lambda w \\ \varepsilon \Delta w^* - \operatorname{div}(\mathbf{v} w^*) - (L(\mathbf{v}, x) - P \cdot \mathbf{v}) w^* = \lambda w^*. \end{cases} \tag{4.14}$$

Multiply the first equation by w^*, the second equation by w, and subtract:

$$\varepsilon (w^* \Delta w - w \Delta w^*) + w^* \mathbf{v} \cdot Dw + w \operatorname{div}(\mathbf{v} w^*) = 0.$$

Observe next that

$$w^* \mathbf{v} \cdot Dw + w \operatorname{div}(\mathbf{v} w^*) = \operatorname{div}(w w^* \mathbf{v}) = \operatorname{div}(\sigma \mathbf{v}) = \operatorname{div}(\sigma D_p H)$$

and

$$w^*\Delta w - w\Delta w^* = -\operatorname{div}\left(w^2 D\left(\frac{w^*}{w}\right)\right) = -\operatorname{div}\left(w^2 D\left(\frac{\sigma}{w^2}\right)\right)$$
$$= -\Delta\sigma + 2\operatorname{div}\left(\sigma\frac{Dw}{w}\right) = -\Delta\sigma + 2\operatorname{div}(\sigma Dv).$$

□

4.3 Second variation. Similarly to the calculations in §3.3, a second variation calculation provides us with estimates on the second derivatives of u^ε:

Theorem 4.3 *We have the estimate*

$$\int_{\mathbb{T}^n} |D^2 u^\varepsilon|^2 \sigma^\varepsilon \, dx \leq C. \tag{4.15}$$

Proof. We drop the superscripts ε, so that the PDE (4.9) becomes

$$\varepsilon\Delta v + \varepsilon|Dv|^2 + H(Du, x) = \lambda.$$

Differentiate twice with respect to x_k:

$$\varepsilon\Delta v_{x_k x_k} + 2\varepsilon v_{x_i x_k} v_{x_i x_k} + 2\varepsilon v_{x_i} v_{x_i x_k x_k} + H_{p_i p_j} u_{x_i x_k} u_{x_i x_k}$$
$$+ H_{p_i} u_{x_i x_k x_k} + 2H_{p_i x_k} u_{x_i x_k} + H_{x_k x_k} = 0.$$

Therefore

$$\int_{\mathbb{T}^n} (H_{p_i p_j} u_{x_i x_k} u_{x_i x_k} + 2\varepsilon v_{x_i x_k} v_{x_i x_k})\sigma \, dx = \tag{4.16}$$
$$-\int_{\mathbb{T}^n} (\varepsilon\Delta v_{x_k x_k} + 2\varepsilon v_{x_i} v_{x_i x_k x_k} + H_{p_i} u_{x_i x_k x_k} + 2H_{p_i x_k} u_{x_i x_k} + H_{x_k x_k})\sigma \, dx.$$

Now the Euler-Lagrange equation (4.13) implies

$$\int_{\mathbb{T}^n} H_{p_i} u_{x_i x_k x_k} \sigma \, dx = -\int_{\mathbb{T}^n} (H_{p_i}\sigma)_{x_i} u_{x_k x_k} \, dx$$
$$= \int_{\mathbb{T}^n} (-\varepsilon\Delta\sigma + 2\varepsilon(\sigma v_{x_i})_{x_i}) u_{x_k x_k} \, dx$$
$$= -\varepsilon\int_{\mathbb{T}^n} \sigma\Delta v_{x_k x_k} \, dx - 2\varepsilon\int_{\mathbb{T}^n} v_{x_i} v_{x_i x_k x_k} \sigma \, dx.$$

Consequently the integral of the first three terms on the right hand side of (4.16) vanishes; whence

$$\int_{\mathbb{T}^n} (H_{p_ip_j} u_{x_ix_k} u_{x_ix_k} + 2\varepsilon v_{x_ix_k} v_{x_ix_k})\sigma\, dx = -\int_{\mathbb{T}^n} (2H_{p_ix_k} u_{x_ix_k} + H_{x_kx_k})\sigma\, dx.$$

□

4.4 Differentiations in the variable P. By analogy with §3.4 we now redefine

$$\bar{H}^\varepsilon(P) := \lambda_\varepsilon = K_\varepsilon[\mathbf{v}_\varepsilon]. \tag{4.17}$$

Theorem 4.4 (i) *We have*

$$D\bar{H}^\varepsilon(P) = \int_{\mathbb{T}^n} \mathbf{v}_\varepsilon \sigma^\varepsilon dx = \int_{\mathbb{T}^n} D_pH(Du^\varepsilon, x)\sigma^\varepsilon dx. \tag{4.18}$$

(ii) *Furthermore,*

$$D^2\bar{H}^\varepsilon(P) = \int_{\mathbb{T}^n} (D_p^2 H D_{xP}u^\varepsilon \otimes D_{xP}u^\varepsilon + 2\varepsilon D_{xP}v^\varepsilon \otimes D_{xP}v^\varepsilon)\sigma^\varepsilon\, dx. \tag{4.19}$$

In particular we see that $P \mapsto \bar{H}^\varepsilon(P)$ is convex.

Proof. 1. As usual, we drop the sub- and superscripts ε. Differentiate the first PDE in (4.14) with respect to P_k:

$$\begin{aligned}\varepsilon\Delta w_{P_k} + \mathbf{v}\cdot Dw_{P_k} + \mathbf{v}_{P_k}\cdot Dw - (L(\mathbf{v},x) - P\cdot\mathbf{v})w_{P_k}&\\ - (D_vL(\mathbf{v},x) - P)\cdot \mathbf{v}_{P_k} w + v^k w &= \lambda w_{P_k} + \lambda_{P_k} w,\end{aligned}$$

where $\mathbf{v} = (v^1, \dots, v^n)$. In view of the identity (4.7), the terms involving $\mathbf{v}_{P_k}$ cancel:

$$\varepsilon\Delta w_{P_k} + \mathbf{v}\cdot Dw_{P_k} - (L(\mathbf{v},x) - P\cdot\mathbf{v})w_{P_k} + v^k w = \lambda w_{P_k} + \lambda_{P_k} w. \tag{4.20}$$

Now multiply by w^* and integrate by parts:

$$\bar{H}_{P_k} = \lambda_{P_k} = \int_{\mathbb{T}^n} v^k w w^*\, dx = \int_{\mathbb{T}^n} v^k \sigma\, dx.$$

2. Upon our dropping the superscripts ε, the PDE (4.9) reads

$$\varepsilon\Delta v + \varepsilon|Dv|^2 + H(Du, x) = \lambda.$$

Differentiate with respect to P_k and then P_l:

$$\varepsilon\Delta v_{P_kP_l} + 2\varepsilon v_{x_iP_k} v_{x_iP_l} + 2\varepsilon v_{x_i} v_{x_iP_kP_l} + H_{p_ip_j} u_{x_iP_k} u_{x_jP_l} + H_{p_i} u_{x_iP_kP_l} = \lambda_{P_kP_l}.$$

Then

$$\bar{H}_{P_kP_l} = \lambda_{P_kP_l} = \int_{\mathbb{T}^n} (H_{p_ip_j} u_{x_iP_k} u_{x_jP_l} + 2\varepsilon v_{x_iP_k} v_{x_iP_l})\sigma\, dx$$
$$+ \int_{\mathbb{T}^n} (\varepsilon \Delta v_{P_kP_l} + 2\varepsilon v_{x_i} v_{x_iP_kP_l} + H_{p_i} u_{x_iP_kP_l})\sigma\, dx.$$

The last integral vanishes, since the Euler-Lagrange equation (4.13) implies

$$\begin{aligned}
\int_{\mathbb{T}^n} H_{p_i} u_{x_iP_kP_l} \sigma\, dx &= -\int_{\mathbb{T}^n} (H_{p_i}\sigma)_{x_i} u_{P_kP_l}\, dx \\
&= \int_{\mathbb{T}^n} (-\varepsilon\Delta\sigma + 2\varepsilon(\sigma v_{x_i})_{x_i}) u_{P_kP_l}\, dx \\
&= -\varepsilon \int_{\mathbb{T}^n} \sigma \Delta v_{P_kP_l}\, dx - 2\varepsilon \int_{\mathbb{T}^n} v_{x_i} v_{x_iP_kP_l} \sigma\, dx.
\end{aligned}$$

□

4.5 Limits as $\epsilon \to 0$. This section mirrors §3.5 by showing that in the limit $\varepsilon \to 0$ the basic structure of weak KAM theory appears.

First note that in view of the PDE (4.9), we have the uniform estimate

$$\sup_{\mathbb{T}^n} |u^\varepsilon|, |Du^\varepsilon| \leq C;$$

and so we may assume upon passing if necessary to a subsequence that the uniform limit

$$\lim_{\varepsilon \to 0} u^\varepsilon =: u$$

exists. As in §3.5 we may also suppose that the limit (3.23) holds for some measure ν and all continuous functions $\Phi = \Phi(p, x)$. We then define μ by the formula (3.24).

Theorem 4.5 (i) *The functions $\bar{H}^\varepsilon$ converge to $\bar{H}$:*

$$\lim_{\varepsilon \to 0} \bar{H}^\varepsilon(P) = \bar{H}(P). \tag{4.21}$$

(ii) *The measure μ satisfies*

$$\int_{\mathbb{R}^n} \int_{\mathbb{T}^n} v \cdot D\phi\, d\mu = 0 \tag{4.22}$$

for all smooth, periodic functions $\phi = \phi(x)$; and

$$\int_{\mathbb{R}^n} \int_{\mathbb{T}^n} v\, d\mu =: V \in \partial \bar{H}(P). \tag{4.23}$$

(iii) *The measure* μ *minimizes the action functional* (2.11) *among all other probability measures satisfying* (4.22) *and* (4.27).

(iv) *The function* u *is a viscosity solution of*

$$-H(Du, x) = -\bar{H}(P). \tag{4.24}$$

Proof. 1. The functions $\bar{H}^\varepsilon$ are convex and locally bounded, and so, passing if necessary to a further subsequence, the limit

$$\lim_{\varepsilon \to 0} \bar{H}^\varepsilon(P) =: K(P) \tag{4.25}$$

exists locally uniformly. Then (4.9) implies $u = P \cdot x + v$ is a viscosity solution of

$$-H(Du, x) = -K(P)$$

and so also solves this PDE almost everwhere. Hence

$$H(Du^\delta, x) \leq K(P) + O(\delta),$$

for $u^\delta := \eta^\delta * u$, $u^\delta = P \cdot x + v^\delta$. We now use v^δ in the variational formula

$$\bar{H}(P) = \inf_{w \in C^1(\mathbb{T}^n)} \sup_{x \in T^n} H(P + Dw, x)$$

and send $\delta \to 0$, to deduce

$$\bar{H}(P) \leq K(P). \tag{4.26}$$

Later we will show that in fact $\bar{H}(P) = K(P)$.

2. As in the proof of Theorem 3.4, we employ the identity (4.13) to show that measure μ satisfies

$$\int_{\mathbb{R}^n} \int_{\mathbb{T}^n} v \cdot D\phi \, d\mu = 0$$

for all smooth, periodic functions $\phi = \phi(x)$. Likewise, (4.25) and (4.18) imply

$$\int_{\mathbb{R}^n} \int_{\mathbb{T}^n} v \, d\mu =: V \in \partial \bar{K}(P). \tag{4.27}$$

Now calculate

$$
\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} L(v,x)\,d\mu + \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} H(p,x)\,d\nu &= \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} L(D_pH,x) + H(p,x)\,d\nu \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} D_pH\cdot p\,d\nu \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{T}^n} D_pH(Du^\varepsilon,x)\cdot Du^\varepsilon\sigma^\varepsilon\,dx \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{T}^n} D_pH(Du^\varepsilon,x)\cdot(P+Dv^\varepsilon)\sigma^\varepsilon\,dx \\
&= P\cdot V + \lim_{\varepsilon\to 0}\int_{\mathbb{T}^n} D_pH(Du^\varepsilon,x)\cdot Dv^\varepsilon\sigma^\varepsilon\,dx.
\end{aligned}
\tag{4.28}
$$

3. Next multiply (4.13) by v^ε and integrate by parts:

$$
\int_{\mathbb{T}^n} D_pH(Du^\varepsilon,x)\cdot Dv^\varepsilon\sigma^\varepsilon\,dx = -\varepsilon\int_{\mathbb{T}^n}(\Delta v^\varepsilon + 2|Dv^\varepsilon|^2)\sigma^\varepsilon\,dx.
\tag{4.29}
$$

Likewise, multiply (4.12) by $w^{\varepsilon *}$ and integrate, to find

$$
\bar{H}^\varepsilon(P) = \int_{\mathbb{T}^n} H(Du^\varepsilon,x)\sigma^\varepsilon\,dx + \int_{\mathbb{T}^n}\varepsilon\Delta w^\varepsilon w^{\varepsilon *}\,dx.
$$

Recall from (4.6) that $v^\varepsilon = \log w^\varepsilon$, and therefore

$$
\Delta w^\varepsilon = (\Delta v^\varepsilon + |Dv^\varepsilon|^2)w^\varepsilon.
$$

We use this identity and the identity before in (4.29), concluding that

$$
\int_{\mathbb{T}^n} D_pH(Du^\varepsilon,x)\cdot Dv^\varepsilon\sigma^\varepsilon\,dx = -\varepsilon\int_{\mathbb{T}^n}|Dv^\varepsilon|^2\sigma^\varepsilon\,dx + \int_{\mathbb{T}^n} H(Du^\varepsilon,x)\sigma^\varepsilon\,dx - \bar{H}^\varepsilon(P).
$$

This identity lets us return to (4.28), to deduce that

$$
\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} L(v,x)\,d\mu \le P\cdot V - K(P) = K^*(V),
$$

where K^* is the dual convex function to K. The last equality holds since (4.27) tells us that $V\in\partial K(P)$. But (4.26) implies for all V that

$$
K^*(V) \le \bar{H}^*(V) = \bar{L}(V).
$$

Therefore

$$
\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} L(v,x)\,d\mu \le \bar{L}(V);
$$

and this fact, as already noted in the proof of Theorem 3.4 means that μ is a minimizing measure. In particular, $\bar{L}\equiv K^*$: whence $K\equiv\bar{H}$. □

5 PDE for deterministic mean field games

Lasry and Lions in [L-L1]–[L-L3] have introduced some fascinating classes of stochastic "mean field games", which in turn correspond to various systems of PDE that generalize the two basic PDE (1.1), (1.2) of weak KAM theory. In the stationary case [L-L2] some of their equations read, in our notation,

$$-\varepsilon\Delta u^\varepsilon + H(Du^\varepsilon, x) = \bar{H}(P) + V[\sigma^\varepsilon]$$

and

$$-\varepsilon\Delta\sigma^\varepsilon + \operatorname{div}(D_pH(Du^\varepsilon, x)\sigma^\varepsilon) = 0,$$

where $\sigma^\varepsilon > 0$ and $\int_{\mathbb{T}^n} \sigma^\varepsilon \, dx = 1$. Here $\varepsilon > 0$ is a viscosity coefficient resulting from the stochastic terms in the dynamics and $V[\cdot]$ is a functional defined for probability measures, the precise form of which depends upon the particular rules of the mean field game. The deterministic version of these two PDE are

$$H(Du, x) = \bar{H}(P) + V[\sigma] \tag{5.1}$$

and

$$\operatorname{div}(D_pH(Du, x)\sigma) = 0. \tag{5.2}$$

These have almost the exact from of our two basic PDE (1.1), (1.2) of weak KAM theory, the only difference being the term $V[\cdot]$ which couples the measure σ with to the Hamilton-Jacobi equation for u. (We note also that coupled PDE of the from (5.1) and (5.2) arise also from WKB expansions for solutions of nonlinear dispersive equations: see for instance Section 14.2 in Whitham [W].)

Let us take

$$V[\sigma] = \Phi(\sigma),$$

for some smooth, nondecreasing function $\Phi : \mathbb{R} \to \mathbb{R}$, and for heuristic purposes assume that u, σ are smooth solutions of (5.1), (5.2).

Theorem 5.1 (i) *We have the estimate*

$$\int_{\mathbb{T}^n} |D^2u|^2\sigma + \Phi'(\sigma)|D\sigma|^2 \, dx \le C. \tag{5.3}$$

(ii) *In particular, if* $\Phi'(s) \ge c|s|^\gamma$ *for positive constants* c *and* γ*, then*

$$\int_{\mathbb{T}^n} \sigma^{\frac{n(\gamma+2)}{n-2}} \, dx \le C. \tag{5.4}$$

The significance is that estimate (5.4) shows $\sigma \in L^p$ for $p = \frac{n(\gamma+2)}{n-2}$. This is not the case for the more weakly coupled PDE (1.1), (1.2) of weak KAM theory, in which σ is in general a measure that may be singular with respect to Lebesgue measure. The term $V[\sigma] = \Phi(\sigma)$ has a regularizing effect, persisting even for the deterministic case $\varepsilon = 0$.

Proof. Differentiate (5.1) twice with respect to x_k:

$$H_{p_i p_j} u_{x_i x_k} u_{x_i x_k} + H_{p_i} u_{x_i x_k x_k} + 2H_{p_i x_k} u_{x_i x_k} + H_{x_k x_k} = (\Phi'(\sigma)\sigma_{x_k})_{x_k}.$$

Next multiply by σ and integrate over $\mathbb{T}^n$, using (5.2) to simplify and obtain the identity

$$\int_{\mathbb{T}^n} H_{p_i p_j} u_{x_i x_k} u_{x_i x_k} \sigma + \Phi'(\sigma)\sigma_{x_k}\sigma_{x_k}\, dx = -\int_{\mathbb{T}^n} (2H_{p_i x_k} u_{x_i x_k} + H_{x_k x_k})\sigma\, dx.$$

The estimate (5.3) follows, and then (5.4) results from the Sobolev inequality. □

6 Nonvariational approximations, "second order" weak KAM theory

In this speculative section we return to the entropy regularization discusses in Section 2, and examine an very singular nonvariational approximation.

6.1 A linearization. To motivate developments in subsequent subsections, we next calculate linearization of the Euler–Lagrange PDE (2.3), which we rewrite in the symmetric form

$$A_H^k[u^k] := -\frac{1}{k} e^{-kH}(e^{kH} H_{p_i})_{x_i} = -\frac{1}{k}(H_{p_i})_{x_i} - H_{p_i} H_{p_j} u^k_{x_i x_j} - H_{x_i} H_{p_i} = 0, \tag{6.1}$$

H evaluated at (Du^k, x). I proved in [E1] that $u = \lim_{k\to\infty} u^k$ is a viscosity solution of Aronsson's equation

$$A_H[u] := -H_{p_i} H_{p_j} u_{x_i x_j} - H_{x_i} H_{p_i} = 0, \tag{6.2}$$

H evaluated at (Du, x).

Define the linearization

$$\begin{aligned} L_k v \;&:=\; \frac{d}{d\tau} A_H^k[u^k + \tau v]\Big|_{\tau=0} \\ &=\; -\frac{1}{k} e^{-kH}(e^{kH} H_{p_i p_j} v_{x_j})_{x_i} - \frac{1}{k} e^{-kH}(e^{kH} k H_{p_j} v_{x_j} H_{p_i})_{x_i} \\ &\qquad + \frac{1}{k} e^{-kH} k H_{p_j} v_{x_j} (e^{kH} H_{p_i})_{x_i}. \end{aligned}$$

Thus

$$L_k v = -\frac{1}{k}(H_{p_ip_j}v_{x_j})_{x_i} - H_{p_ip_j}(H)_{x_i}v_{x_j} - H_{p_i}(H_{p_j}v_{x_j})_{x_i}. \tag{6.3}$$

Again $H = H(Du^k, x)$. The formal adjoint of L_k with respect to the usual L^2 inner product is

$$L_k^* w := -\frac{1}{k}(H_{p_ip_j}w_{x_i})_{x_j} + (H_{p_ip_j}(H)_{x_i}w)_{x_j} - (H_{p_j}(H_{p_i}w)_{x_i})_{x_j}. \tag{6.4}$$

Observe that $\sigma^k = e^{k(H-\bar{H}^k)}$ solves

$$L_k^*\sigma^k = 0, \tag{6.5}$$

since $(\sigma^k H_{p_i})_{x_i} = 0$ and $\sigma^k_{x_i} = \sigma^k k(H)_{x_i}$. We will return to this observation in the next section.

6.2 A solution for the dual operator, symmetry. Define now the weighted inner products

$$\langle f, g\rangle_k := \int_{\mathbb{T}^n} fg\sigma^k dx, \quad [f,g]_k := \int_{\mathbb{T}^n} fg(\sigma^k)^{-1}dx.$$

We show next that L_k is symmetric with respect to$\langle\cdot,\cdot\rangle_k$, and L_k^* is symmetric with respect to$[\cdot,\cdot]_k$:

Theorem 6.1 (i) *For all smooth v, w, we have*

$$\langle L_k v, w\rangle_k = \langle v, L_k w\rangle_k = \int_{\mathbb{T}^n}\left\{(H_{p_i}v_{x_i})(H_{p_j}w_{x_j}) + \frac{1}{k}H_{p_ip_j}v_{x_j}w_{x_i}\right\}\sigma^k\,dx. \tag{6.6}$$

As usual, H is evaluated at (Du^k, x).

(ii) *Furthermore*

$$L_k^*(\sigma^k w) = \sigma^k L_k w \tag{6.7}$$

and

$$\begin{aligned} [L_k^* v, w]_k &= [v, L_k^* w]_k \\ &= \textstyle\int_{\mathbb{T}^n}\left\{\left(H_{p_i}\left(\frac{v}{\sigma^k}\right)_{x_i}\right)\left(H_{p_j}\left(\frac{w}{\sigma^k}\right)_{x_j}\right) + \frac{1}{k}H_{p_ip_j}\left(\frac{v}{\sigma^k}\right)_{x_i}\left(\frac{w}{\sigma^k}\right)_{x_j}\right\}\sigma^k\,dx. \end{aligned} \tag{6.8}$$

Proof. 1. Define for $\tau \in \mathbb{R}$ the inner product

$$\langle f, g\rangle_\tau := \int_{\mathbb{T}^n} fge^{kH(Du^k+\tau Dv,x)}dx.$$

Then

$$(6.9) \qquad \langle A_H^k[u^k+\tau v], w\rangle_\tau = -\frac{1}{k}\int_{\mathbb{T}^n} e^{-kH}(e^{kH}H_{p_i})_{x_i} w e^{kH}\,dx = \frac{1}{k}\int_{\mathbb{T}^n} H_{p_i} w_{x_i} e^{kH} dx,$$

H evaluated at $(Du^k+\tau Dv, x)$. We compute

$$\frac{d}{d\tau}\langle A_H^k[u^k+\tau v], w\rangle_\tau\bigg|_{\tau=0} = \langle L_k v, w\rangle_0 + \langle A_H^k[u^k], wkD_pH\cdot Dv\rangle_0 = \langle L_k v, w\rangle_0,$$

since $A_H^k[u^k]\equiv 0$. Also,

$$\frac{d}{d\tau}\left(\frac{1}{k}\int_{\mathbb{T}^n} H_{p_i}w_{x_i}e^{kH}dx\right)\bigg|_{\tau=0} = \int_{\mathbb{T}^n}\left(\frac{1}{k}H_{p_ip_j}v_{x_j}w_{x_i} + (H_{p_i}w_{x_i})(H_{p_j}v_{x_j})\right)e^{kH}dx.$$

Insert the previous two calculations into (6.9) and multiply by $e^{-k\bar{H}^k}$ to obtain (6.6).

2. We have

$$\int_{\mathbb{T}} vL_k^*(\sigma^k w)\,dx = \int_{\mathbb{T}^n} L_k v w\sigma^k\,dx = \int_{\mathbb{T}^n} v(L_k w)\sigma^k\,dx,$$

for all smooth functions v and w. Hence (6.7) holds, and consequently

$$L_k^* w = \sigma^k L_k\left(\frac{w}{\sigma^k}\right).$$

Then

$$[L_k^* v, w]_k = \left[\sigma^k L_k\left(\frac{v}{\sigma^k}\right), w\right]_k = \left(L_k\left(\frac{v}{\sigma^k}\right), w\right) = \left(\frac{v}{\sigma^k}, L_k^* w\right) = [v, L_k^* w]_k.$$

Also

$$\begin{aligned}[L_k^* v, w]_k &= \left(L_k\left(\frac{v}{\sigma^k}\right), w\right) = \left\langle L_k\left(\frac{v}{\sigma^k}\right), \frac{w}{\sigma^k}\right\rangle_k \\ &= \int_{\mathbb{T}^n}\left\{\left(H_{p_i}\left(\frac{v}{\sigma^k}\right)_{x_i}\right)\left(H_{p_j}\left(\frac{w}{\sigma^k}\right)_{x_j}\right) + \frac{1}{k}H_{p_ip_j}\left(\frac{v}{\sigma^k}\right)_{x_i}\left(\frac{w}{\sigma^k}\right)_{x_j}\right\}\sigma^k dx.\end{aligned}$$

□

6.3 A nonvariational approximation. Recall from the introduction that the PDE approach to weak KAM turns upon the pair of equations (1.1) and (1.2). The variational approximation (2.1) replaces (1.1) with Aronsson's PDE

$$(1.1)' \qquad A_H[u] := -H_{p_i}(Du,x)H_{p_j}(Du,x)u_{x_ix_j} - H_{x_i}(Du,x)H_{p_i}(Du,x) = 0.$$

I suggest now that the proper generalization of (1.2) is the PDE

$$(1.2)' \qquad L\sigma := -(H_{p_j}(Du,x)(H_{p_i}(Du,x)\sigma)_{x_i})_{x_j} = 0,$$

and interpret the pair of PDE (1.1)$'$, (1.2)$'$ as a "second order" version of weak KAM theory. Note carefully that sufficiently smooth solutions u, σ of (1.1) and (1.2) also solve (1.1)$'$, (1.2)$'$. However a solution u of (1.1)$'$ need not solve (1.1), and it is not clear to me if a solution of (1.2)$'$ must necessarily solve (1.2).

We conclude with the derivation of some uniform estimates for a regularization of the foregoing PDE. That such natural estimates hold is perhaps an indication that (1.2)$'$ is indeed a proper generalization of (1.2). Consider for $\delta > 0$ the equation

$$
\begin{aligned}
A_H^\delta[u^\delta] &= -\delta\Delta u^\delta + A_H[u^\delta] \\
&= -\delta\Delta u^\delta - H_{p_i}(Du^\delta, x)H_{p_j}(Du^\delta, x)u^\delta_{x_i x_j} - H_{p_i}(Du^\delta, x)H_{x_i}(Du^\delta, x) \\
&= 0
\end{aligned}
\tag{6.10}
$$

and the corresponding linearization

$$
L_\delta v := \frac{d}{d\tau} A_H^\delta[u^\delta + \tau v]\bigg|_{\tau=0}.
$$

A calculation shows

$$
L_\delta v = -\delta\Delta V - H_{p_j}(H_{p_i} v_{x_i})_{x_j} - H_{p_i p_j}(H)_{x_i} v_{x_j}, \tag{6.11}
$$

H evaluated at (Du^δ, x). The adjoint of L_δ with respect to the usual L^2 inner product is

$$
L_\delta^* w = -\delta\Delta w - (H_{p_j}(H_{p_i} w)_{x_i})_{x_j} + (H_{p_i p_j}(H)_{x_i} w)_{x_j}. \tag{6.12}
$$

Next, let $\sigma^\delta > 0$ solve

$$
L_\delta^* \sigma^\delta = -\delta\Delta\sigma^\delta - (H_{p_j}(H_{p_i}\sigma^\delta)_{x_i})_{x_j} + (H_{p_i p_j}(H)_{x_i}\sigma^\delta)_{x_j} = 0, \tag{6.13}
$$

with the normalization

$$
\int_{\mathbb{T}^n} \sigma^\delta \, dx = 1.
$$

We conclude our paper with the following elegant energy estimate, which is an analog of (3.10) and (4.15):

Theorem 6.2 *We have the estimate*

$$
\int_{\mathbb{T}^n} \left(\frac{|DH|^2}{\delta} + |D^2 u^\delta|^2 \right) \sigma^\delta \, dx \le C \tag{6.14}
$$

for a constant C independent of $\delta > 0$.

Proof. Multiply (6.13) by $H = H(Du^\delta, x)$ and integrate by parts. We calculate

$$\begin{aligned}
\int_{\mathbb{T}^n} H_{p_i p_j} (H)_{x_i} (H)_{x_j} \sigma^\delta dx &= \int_{\mathbb{T}^n} \delta \sigma^\delta_{x_i} (H)_{x_i} + H_{p_j} (H)_{x_j} (H_{p_i} \sigma^\delta)_{x_i} \, dx \\
&= \int_{\mathbb{T}^n} \delta \sigma^\delta_{x_i} (H)_{x_i} - \delta \Delta u^\delta (H_{p_i} \sigma^\delta)_{x_i} \, dx \\
&= \delta \int_{\mathbb{T}^n} \sigma^\delta_{x_i} (H_{p_k} u^\delta_{x_k x_i} + H_{x_i}) - u^\delta_{x_i x_j} (H_{p_i} \sigma^\delta)_{x_j} \, dx \\
&= \delta \int_{\mathbb{T}^n} \sigma^\delta_{x_i} H_{x_i} - u^\delta_{x_i x_j} (H_{p_i})_{x_j} \sigma^\delta \, dx \\
&= -\delta \int_{\mathbb{T}^n} ((H_{x_i})_{x_i} + H_{p_i x_j} u^\delta_{x_i x_j} + H_{p_i p_k} u^\delta_{x_i x_j} u^\delta_{x_k x_j}) \sigma^\delta \, dx.
\end{aligned}$$

This identity leads to (6.14). $\square$

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