Partial Differential Equations

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1 Overview

This article is an extremely rapid survey of the modern theory of partial differential equations (PDEs). Sources of PDEs are legion: mathematical physics, geometry, probability theory, continuum mechanics, optimization theory, etc. Indeed, *most of the fundamental laws of the physical sciences are partial differential equations* and most papers published in applied math concern PDEs.

The following discussion is consequently very broad, but also very shallow, and will certainly be inadequate for any given PDE the reader may care about. The goal is rather to highlight some of the many key insights and unifying principles across the entire subject.

1.1 Confronting PDEs

Among the greatest accomplishments of the physical and other sciences are the discoveries of fundamental laws, usually partial differential equations. The great problems for mathematicians, both pure and applied, are then to understand the solutions of these equations, using theoretical analysis, numerical simulations, perturbation theory, and whatever other tools we can find.

But this very success in physics, that some fairly simple looking PDEs, for example the Euler equations for fluid mechanics (see (11) below), model very complicated and diverse physical phenomena, causes all sorts of mathematical difficulties. Whatever general assertion we try to show mathematically must apply to all sorts of solutions with extremely disparate behavior.

It is therefore a really major undertaking to understand solutions of partial differential equations, and for this there are at least three primary mathematical approaches:

- *discovering analytical formulas for solutions, either exact or approximate*,
- *devising accurate and fast numerical methods*, and
- *developing rigorous theory.*

In other words, we can aspire to actually solve the PDE more-or-less explicitly, to compute solutions, or else to indirectly deduce properties of the solutions (without relying upon formulas or numerics). This article surveys these viewpoints, with particular emphasis upon the last.

Terminology. A *partial differential equation* (PDE) is an equation involving an unknown function u of more than one variable and certain of its partial derivatives. The *order* of a PDE is the order of the highest order partial derivative of the unknown appearing within it.

A *system* of PDEs comprises several equations involving an unknown vector-valued function $\mathbf{u}$ and its partial derivatives.

A PDE is *linear* if it corresponds to a linear operator acting on the unknown and its partial derivatives; otherwise the partial differential equation is *nonlinear*.

Notation. Hereafter u usually denotes the real-valued solution of a given PDE, and is usually a function of points $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, typically denoting a position in space, and sometimes also a function of $t \in \mathbb{R}$, denoting time. We write $u_{x_k} = \frac{\partial u}{\partial x_k}$ to denote the partial derivative of u with respect to x_k, $u_t = \frac{\partial u}{\partial t}$, $u_{x_k x_l} = \frac{\partial^2 u}{\partial x_k \partial x_l}$, etc for higher partial derivatives. The *gradient* of u in the variable x is

$$\nabla u = (u_{x_1}, \dots, u_{x_n}).$$

(In this article, ∇u always as above denotes the gradient in the variables x, even if u also depends on t.) We write the *divergence* of a vector field $\mathbf{F} = (F^1, \dots, F^n)$ as $\operatorname{div} \mathbf{F} = \sum_{i=1}^n F^i_{x_i}$.

The *Laplacian* of u is the divergence of its gradient:

$$\Delta u = \nabla^2 u = \textstyle\sum_{k=1}^n u_{x_k x_k}. \tag{1}$$

Let us also write $\mathbf{u} = (u^1, \dots, u^m)$ to display the components of a vector-valued function. We always use boldface for vector-valued mappings.

The solid n-dimensional ball with center x and radius r is denoted $B(x, r)$, and $\partial B(x, r)$ is its boundary, a sphere. More generally, ∂U means the boundary of a set $U \subset \mathbb{R}^n$; and we denote by

$$\int_{\partial U} f \, dS$$

the integral of a function f over the boundary, with respect to $(n-1)$-dimensional surface area.

1.2 Some important partial differential equations

Following is a listing of some of the most commonly studied PDEs. To streamline and clarify the presentation, we have mostly set various physical parameters to unity in these equations.

1.2.1 First-order PDEs

First-order PDEs appear in many physical theories, mostly in dynamics, continuum mechanics and optics. For example, in the *scalar conservation law*

$$u_t + \operatorname{div} \mathbf{F}(u) = 0 \tag{2}$$

the unknown u is the density of some physically interesting quantity and the vector field $\mathbf{F}(u)$, its flux, depends nonlinearly upon u.

Another important first-order PDE, the *Hamilton-Jacobi equation*

$$u_t + H(\nabla u, x) = 0, \tag{3}$$

appears in classical mechanics and in optimal control theory. In these contexts, H is called the *Hamiltonian*.

1.2.2 Second-order PDEs

Second-order PDEs model a significantly wider variety of physical phenomena than do first-order equations. For example, among its many other interpretations, *Laplace's equation*

$$\Delta u = 0 \tag{4}$$

records diffusion effects in equilibrium. Its time dependent analog is the *heat equation*

$$u_t - \Delta u = 0, \tag{5}$$

also known as the *diffusion equation*.

The *wave equation*

$$u_{tt} - c^2 \Delta u = 0 \tag{6}$$

superficially somewhat resembles the heat equation, but as the name suggests supports solutions with utterly different behavior.

Schrödinger's equation

$$iu_t + \Delta u = 0, \tag{7}$$

for which solutions u are complex-valued, is the quantum mechanics analog of the wave equation.

1.2.3 Systems of PDEs

In a *system of conservation laws*

$$\mathbf{u}_t + \operatorname{div} \mathbf{F}(\mathbf{u}) = \mathbf{0}, \tag{8}$$

each component of $\mathbf{u} = (u^1, \dots, u^m)$ typically represents a mass, momentum or energy density.

A *reaction-diffusion system* of partial differential equations has the form

$$\mathbf{u}_t - \Delta \mathbf{u} = \mathbf{f}(\mathbf{u}). \tag{9}$$

Here the components of $\mathbf{u}$ typically represent densities of, say, different chemicals, whose interactions are modeled by the nonlinear term $\mathbf{f}$.

The simplest form of *Maxwell's equations* reads

$$\begin{cases} \mathbf{E}_t = \operatorname{curl} \mathbf{B} \\ \mathbf{B}_t = -\operatorname{curl} \mathbf{E} \\ \operatorname{div} \mathbf{E} = \operatorname{div} \mathbf{B} = 0, \end{cases} \tag{10}$$

in which $\mathbf{E}$ is the electric field and $\mathbf{B}$ the magnetic field.

Fluid mechanics provides some of the most complicated and fascinating systems of PDEs in applied mathematics. The most important are *Euler's equations for incompressible, inviscid fluid flow*:

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \tag{11}$$

and the *Navier–Stokes equations for incompressible, viscous flow*:

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} = -\nabla p \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \tag{12}$$

In these systems $\mathbf{u}$ denotes the fluid velocity and p the pressure.

1.2.4 Higher order PDEs

Equations of order greater than two are much less common. Generally speaking, such higher order PDEs do not represent fundamental physical laws, but are rather derived from such.

For instance, we can sometimes rewrite a system of two second–order equations as a single

fourth–order PDE. In this way the *biharmonic equation*

$$\Delta^2 u = 0 \tag{13}$$

comes up in linear elasticity theory.

The *Korteweg–deVries* (KdV) *equation*

$$u_t + auu_x + bu_{xxx} = 0, \tag{14}$$

a model of shallow water waves, similarly appears upon our combining a complicated system of lower order equations appearing in appropriate asymptotic expansions.

1.3 Boundary and initial conditions

Partial differential equations very rarely appear alone: most problems require us to solve the PDEs subject to appropriate boundary and/or initial conditions. If for instance we are to study a solution $u = u(x)$, defined for points x lying in some region $U \subset \mathbb{R}^n$, we usually prescribe also something about how u behaves on the boundary ∂U. Most common are *Dirichlet's boundary condition*

$$u = 0 \quad \text{on } \partial U \tag{15}$$

and *Neuman's boundary condition*

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial U, \tag{16}$$

where ν denotes the outward-pointing unit normal to the boundary and $\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$ is the *outer normal derivative*. If, say, u represents a temperature, then (15) specifies that the temperature is held constant on the boundary, and (16) that the heat flux through the boundary is zero.

Imposing initial conditions is usually appropriate for time-dependent PDEs, for which we require for the solution $u = u(x,t)$ that

$$u(\cdot, 0) = g, \tag{17}$$

where $g = g(x)$ is a given function, comprising the *initial data*. For partial differential equations that are second order in time, such as the wave equation (6), it is usually appropriate to specify also

$$u_t(\cdot, 0) = h. \tag{18}$$

2 Understanding PDEs

We explore in this section several general procedures for understanding partial differential equations and their solutions.

2.1 Exact solutions

The most effective approach is of course just to solve the PDE outright, if we can. For instance the boundary–value problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0,1) \\ u = g & \text{on } \partial B(0,1) \end{cases}$$

is solved by *Poisson's formula*

$$u(x) = \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}\, dS,$$

$\alpha(n)$ denoting the volume of the unit ball in $\mathbb{R}^n$.

The solution of the initial-value problem for the wave equation in one space dimension,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

is provided by *d'Alembert's formula*

$$u(x,t) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\, dy. \tag{19}$$

The wave equation can also be solved in higher dimensions, but the formulas become increasingly complicated. For example, *Kirchhoff's formula*

$$u(x,t) = \frac{1}{4\pi c^2 t} \int_{\partial B(x,ct)} h\, dS + \frac{\partial}{\partial t} \left\{ \frac{1}{4\pi c^2 t} \int_{\partial B(x,ct)} g\, dS \right\} \tag{20}$$

satisfies this initial-value problem for wave equation in 3 space dimensions:

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{21}$$

The initial-value problem for the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{22}$$

has for all dimensions the explicit solution

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy. \qquad (23)$$

Certain nonlinear PDEs, including the KdV equation (14), are also exactly solvable; and discovering these so-called *integrable partial differential equations* is a very important undertaking.

It is however a fundamental truth that we cannot solve most partial differential equations, if by "solve" we mean coming up with a more-or-less explicit formula for the answer.

2.2 Approximate solutions, perturbation methods

It is consequently important to realize that *we can often deduce properties of solutions without actually solving the partial differential equation*, either explicitly or numerically.

One such approach develops systematic perturbation schemes to build small "corrections" to a known solution. There is a vast repertoire of such techniques. Given a PDE depending on a small parameter ϵ, the idea is to posit some form for the corrections and to plug this guess into the differential equation, trying then to fine tune the form of the perturbations to make the error as small as possible. These procedures do not usually amount to proofs, but rather construct *self-consistent* guesses.

Multiple scales. *Homogenization* problems entail PDEs with effects occurring on differing spatial or temporal scales, say of respective orders 1 and ϵ. Often a goal is to derive simpler *effective PDE* that yield good approximations. We guess the form of the effective equations by supposing an asymptotic expansion of the form

$$u^\epsilon(x) \sim \sum_{k=0}^{\infty} \epsilon^k u^k(x, x/\epsilon)$$

and showing that the leading term u^0 is a function of x alone, solving some kind of simpler equation.

This example illustrates the insight that *simpler behavior often appears in asymptotic limits.*

Asymptotic matching. Solution of PDEs often display differing properties in different subregions. In this circumstance we can try to fashion an approximate solution by first (a) constructing simpler approximate solutions in each subregion and then (b) appropriately *matching* these solutions across areas of overlap.

A common such application is to *boundary layers*. The *outer expansion* for the solution within some region often has a form like

$$u^\epsilon(x) \sim \sum_{k=0}^{\infty} \epsilon^k u^k(x). \qquad (24)$$

Suppose we expect different behavior near the boundary, which we take for simplicity to be the plane $\{x_n = 0\}$. We can then introduce the *stretched variables* $y_n = x_n/\epsilon^\alpha$, $y_i = x_i\, (i = 1, \ldots, n-1)$ and put $\bar{u}^\epsilon(y) = u^\epsilon(x)$. We look then for an *inner expansion*

$$\bar{u}^\epsilon(y) \sim \sum_{k=0}^{\infty} \epsilon^k \bar{u}^k(y). \qquad (25)$$

The idea now is to match terms in the outer expansion (24) in the limit $x_n \to 0$ with terms in the inner expansion (25) in the limit $y_n \to \infty$. Working this out determines for instance the value of α in the scaling.

2.3 Numerical analysis of PDEs

Devising effective computer algorithms for PDEs is a vast enterprise, far beyond the scope of this article; and great ingenuity has gone into the design and implementation of such methods.

Among the most popular are the *finite difference methods* (which approximate functions by values at grid points), the *method of lines* (which discretizes all but the time variable), the *finite element method* and *spectral methods* (which represents functions using carefully designed basis functions), *multigrid methods* (which employ discretizations across different spatial scales), the *level set method* (which represents free boundaries as a level set of a function), and many other schemes.

The design and analysis of such useful NUMERICAL METHODS [IV.29], especially for nonlinear equations, depends upon a good theoretical understanding of the underlying PDE.

2.4 Theory and the importance of estimates

The fully rigorous theory of PDEs focuses largely upon the foundational issues of the existence, smoothness, and, where appropriate, uniqueness of solutions. Once these issues are resolved, at least provisionally, theorists turn attention to understanding the behavior of solutions.

A key point is availability, or not, of strong analytic estimates. Many physically relevant PDEs predict that various quantities are conserved, but these identities are usually not strong enough to be useful, especially in three dimensions. For nonlinear PDEs the higher derivatives solve increasingly complicated, and thus intractable, equations. And so a major dynamic in modern theory is the interplay between (a) deriving "hard" analytic estimates for PDEs and (b) devising "soft" mathematical tools to exploit these estimates. In the remainder of this article, we present for many important PDEs the key estimates upon which rigorous mathematical theory is built.

3 Behavior of solutions

Since PDEs model so vast a range of physical and other phenomena, their solutions display an even vaster range of behaviors. But some of these are more prevalent than others.

3.1 Waves

Many partial differential equations of interest in applied mathematics support at least some solutions displaying "wavelike" behavior.

The wave equation. The wave equation is of course an example, as is most easily seen in one space dimension from d'Alembert's formula (19). This dictates that the solution has the general form $u(x,t) = F(x+ct) + G(x-ct)$ and consequently is the sum of right- and left-moving waves with speed c. The wave-like behavior encoded within Kirchhoff's formula (20) in 3 space dimensions is somewhat less obvious.

Traveling waves. A solution u of a PDE involving time t and the single space variable $x \in \mathbb{R}$ is a *traveling wave* if it has the form

$$u(x,t) = v(x - \sigma t) \tag{26}$$

for some *speed* σ. More generally, a solution u of a PDE in more space variables having the form

$$u(x,t) = v(y \cdot x - \sigma t)$$

is a *plane wave*. An extremely useful first step for studying a PDE is to look for solutions with these special structures.

Dispersion. It is often informative to look for plane wave solutions of the complex form

$$u(x,t) = e^{i(y\cdot x - \sigma t)} \tag{27}$$

where $\sigma \in \mathbb{C}$ and $y \in \mathbb{R}^n$. We plug the guess (27) into some given linear PDE, thereby to discover the so-called *dispersion relationship* between y and $\sigma = \sigma(y)$ forced by the algebraic structure.

For example, inserting (27) into the *Klein–Gordon equation*

$$u_{tt} - \Delta u + m^2 u = 0 \tag{28}$$

gives $\sigma = \pm(|y|^2 + m^2)^{\frac{1}{2}}$. Hence the speed $\frac{\sigma}{|y|}$ of propagation depends nonlinearly upon the frequency of the initial data $e^{iy\cdot x}$. So *waves of different frequencies propagate at different speeds*; hence the dispersion.

Solitons. As a nonlinear example, putting (26) into the KdV equation (14) with $a = 6$, $b = 1$ leads to the ODE

$$-\sigma v' + 6vv' + v''' = 0,$$

a solution of which is the explicit profile

$$v(s) = \frac{\sigma}{2}\mathrm{sech}^2\left(\frac{\sqrt{\sigma}}{2}s\right)$$

for each speed σ. The corresponding traveling wave $u(x,t) = v(x - \sigma t)$ is called a *soliton*.

3.2 Diffusion and smoothing

From the explicit formula (23) we can read off a lot of interesting quantitative information about the solution u of the initial-value problem (22) for the heat equation.

In particular, notice from (23) that if the initial data function g is merely integrable, the solution u is infinitely differentiable in both the variables x and t at later times. So *the heat equation instantly smooths its initial data*; this observation makes sense as the PDE models diffusive effects.

3.3 Propagation speeds

It is also easy to deduce from (23) that if u solves the heat equation, then values of the initial data $g(y)$ at all points $y \in \mathbb{R}^n$ contribute to determining the solution at (x,t) for times $t > 0$. We can interpret this as an "infinite propagation speed" phenomenon.

By contrast, for many time-dependent PDEs we have "finite propagation speed": there is no influence of some of initial data upon the solution until enough time passes. This is so for first-order PDE in general, for the wave equation, and remarkably also for some nonlinear diffusion PDEs, such as the *porous medium equation*

$$u_t - \Delta(u^\gamma) = 0 \tag{29}$$

with $\gamma > 1$. The particular explicit solution

$$u(x,t) = \frac{1}{t^\alpha}\left(b - \frac{\gamma-1}{2\gamma}\beta\frac{|x|^2}{t^{2\beta}}\right)_+^{\frac{1}{\gamma-1}} \tag{30}$$

for $\alpha = \frac{n}{n(\gamma-1)+2}$, $\beta = \frac{1}{n(\gamma-1)+2}$ and $x_+ = \max\{x, 0\}$ shows clearly that the region of positivity moves outward at finite speed.

3.4 Pattern formation

Nonlinear terms can interplay with diffusion and create interesting effects. For example let $\Phi(z) = \frac{1}{4}(z^2-1)^2$ denote a "two well" potential, having minima at $z = \pm 1$. Look now at this scalar reaction-diffusion problem in which $\epsilon > 0$ is a small parameter:

$$\begin{cases} u_t^\epsilon - \Delta u^\epsilon = \frac{1}{\epsilon^2}\Phi'(u^\epsilon) & \text{in } \mathbb{R}^2 \times (0,\infty) \\ u^\epsilon = g^\epsilon & \text{on } \mathbb{R}^2 \times \{t=0\}. \end{cases}$$

For suitable designed initial data functions g^ϵ, it turns out that that

$$\lim_{\epsilon \to 0} u^\epsilon(x,t) = \pm 1;$$

so that the solution asymptotically goes to one or the other of the two minima of Φ. We can informally think of these regions as colored black and white.

For each time $t \geq 0$, denote by $\Gamma(t)$ the curve between the regions $\{u^\epsilon(\cdot,t) \to 1\}$ and $\{u^\epsilon(\cdot,t) \to -1\}$. Asymptotic matching methods reveal that *the normal velocity of* $\Gamma(t)$ *equals its curvature.* This is a *geometric law of motion* for the evolving black/white patterns emerging in the asymptotic limit $\epsilon \to 0$.

Much more complex pattern formation effects can be modeled by systems of reaction-diffusion PDEs of the general form (9): see PATTERN FORMATION [X.YY],

3.5 Blow-up

Solutions of time-dependent PDEs may or may not exist for all future times, even if their initial conditions at time $t = 0$ are well behaved. Note for example that among solutions of the nonlinear heat equation

$$u_t - \Delta u = u^2, \tag{31}$$

subject to Neumann boundary conditions (16), are those solutions $u = u(t)$ that do not depend on x and consequently that solve the ODE $u_t = u^2$. It is not hard to show that solutions of this equation go to infinity ("blow up") at a finite positive time, if $u(0) > 0$.

For more general initial data, there is an interesting competition between the diffusive, and so stabilizing, term Δu and the destabilizing term u^2.

3.6 Shocks

Solutions of PDEs can fail to exist for large times for other reasons than blow up in the sup-norm. Another possibility is that the gradient of a solution becomes singular at some finite time.

This effect occurs for conservation laws (2). Consider for example this initial-value problem for *Burgers' equation*:

$$\begin{cases} u_t + \frac{1}{2}(u^2)_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{32}$$

Assume we have a smooth solution u and define the *characteristic curve* $x(t)$ to solve the ODE

$$\begin{cases} \dot{x}(t) = u(x(t),t) & (t \geq 0) \\ x(0) = x_0. \end{cases}$$

Then

$$\frac{d}{dt}u(x(t),t) = u_x(x(t),t)\dot{x}(t) + u_t(x(t),t)$$
$$= u_x(x(t),t)u(x(t),t) + u_t(x(t),t) = 0,$$

according to the PDE (32). Consequently $u(x(t),t) \equiv g(x_0)$ and also the characteristic "curve" $x(t)$ is in fact a straight line.

So far, so good; and yet the foregoing often implies that *the PDE does not in fact possess a smooth solution*, existing for all times. For we can easily build initial data g for which the characteristic lines emanating from two distinct initial points cross at a later time.

A major task for the rigorous analysis of Burgers' equation and related conservation laws is characterizing surfaces of discontinuity (called *shocks*) for appropriately defined generalized solutions.

3.7 Free boundaries

Some very difficult problems require not only finding the solution of some PDE, but also the region within which it holds. Consider for example the *Stefan problem*, which asks us to determine the temperature within some body of water surrounded by ice. The temperature solves the heat equation inside a region whose shape changes in time as the ice melts and/or the water freezes. The unknowns are therefore both the temperature profile and the so-called *free boundary* of the water.

There are in general two sorts of such free boundary problems that occur in PDE theory: those for which the free boundary is explicit, such as the Stefan problem, and those for which it is implicit. An example of the latter is the *obstacle problem*:

$$\min\{u, -\Delta u - f\} = 0.$$

The free boundary is

$$\Gamma = \partial\{u > 0\},$$

along which the solution satisfies the *overdetermined boundary conditions* $u = 0$, $\frac{\partial u}{\partial \nu} = 0$. Many important physical and engineering free boundary problems can be cast as obstacle problems.

Much more complicated free boundary problems occur in fluid mechanics, in which the unknown velocity $\mathbf{u}$ satisfies differing sorts of PDE within the *sonic* and *subsonic* regions. We say that the equations *change type* across the free boundary.

4 Some technical methods

So vast is the field of partial differential equations that no small handful of procedures can possibly handle them all. Rather, mathematicians have discovered over the years, and continue to discover, all sorts of useful technical devices and tricks. This section provides a selection of some of the most important.

4.1 Transform methods

A panoply of integral transforms is available to convert linear, constant coefficient PDEs into algebraic equations. The most important is the Fourier transform [I.19]

$$\hat{u}(y) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x)\, dx.$$

Consider, as an example, the equation

$$-\Delta u + u = f \quad \text{in } \mathbb{R}^n. \tag{33}$$

We apply the Fourier transform and learn that $(1 + |y|^2)\hat{u} = \hat{f}$. This algebraic equation lets us easily find $\hat{u}$, after which a somewhat tricky inversion yields the formula

$$u(x) = \frac{1}{(4\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-s-\frac{|x-y|^2}{4s}}}{s^{n/2}} f(y)\, dyds.$$

Strongly related are *Fourier series* methods, which represent solutions of certain PDEs on bounded domains as infinite sums entailing sines and cosines. Another favorite is the *Laplace transform*, which for PDEs is mostly useful as a transform in the time variable.

4.2 Energy methods; functional analytic framework

For many PDEs various sorts of "energy estimates" are valid, where we use this term loosely

to mean integral expressions involving squared quantities.

Integration by parts. Important for what follows is the *integration by parts formula*:

$$\int_U u_{x_i} v \, dx = -\int_U u v_{x_i} \, dx + \int_{\partial U} u v \nu^i \, dS$$

for each $i = 1, \dots, n$. Here ν denotes the outward-pointing unit normal to the boundary. This is a form of the DIVERGENCE THEOREM [X.YY] from multivariable calculus.

4.2.1 Energy estimates

Assume that u solves *Poisson's equation*

$$-\Delta u = f \quad \text{in } \mathbb{R}^n. \tag{34}$$

Then, assuming u goes to zero as $|x| \to \infty$ fast enough to justify the integration by parts, we compute that

$$\int_{\mathbb{R}^n} f^2 \, dx = \int_{\mathbb{R}^n} \sum_{i,j=1}^n u_{x_i x_i} u_{x_j x_j} \, dx$$
$$= -\int_{\mathbb{R}^n} \sum_{i,j=1}^n u_{x_i x_i x_j} u_{x_j} \, dx = \int_{\mathbb{R}^n} \sum_{i,j=1}^n (u_{x_i x_j})^2 \, dx.$$

This identity implies something remarkable: if the Laplacian Δu (which is the sum of the pure second derivatives $u_{x_i x_i}$ for $i = 1, \dots, n$) is square-integrable, then each individual second derivative $u_{x_i x_j}$ for $i, j = 1, \dots, n$ is square-integrable, even those mixed second derivatives that do not even appear in the equation (34).

This is an example of *regularity theory*, which aims to deduce the higher integrability and/or smoothness properties of solutions.

4.2.2 Time dependent energy estimates

As a next example suppose $u = u(x,t)$ solves the wave equation (6) and define the *energy* at time t:

$$e(t) := \frac{1}{2} \int_{\mathbb{R}^n} \left(u_t^2 + c^2 |\nabla u|^2 \right) \, dx.$$

Then, assuming u goes to zero as $|x| \to \infty$ fast enough, we have

$$\dot{e}(t) = \int_{\mathbb{R}^n} \left(u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t \right) \, dx$$
$$= \int_{\mathbb{R}^n} u_t (u_{tt} - c^2 \Delta u) \, dx = 0;$$

This demonstrates *conservation of energy*.

For the *nonlinear wave equation*

$$u_{tt} - \Delta u + f(u) = 0 \tag{35}$$

a similar calculation works for the modified energy

$$e(t) = \int_{\mathbb{R}^n} \left(\frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 + F(u) \right) \, dx,$$

where $f = F'$.

4.3 Variational problems

By far the most successful of the nonlinear theories is *the calculus of variations*; and indeed a fundamental question to ask of any given PDE is whether or not it is *variational*, meaning that it appears as follows.

Given the *Lagrangian density function* $L = L(v, z, x)$, we introduce the functional

$$I[u] := \int_U L(\nabla u, u, x) \, dx,$$

defined for functions $u : U \to \mathbb{R}$, subject to given boundary conditions that are not specified here. Suppose hereafter that u is a *minimizer* of $I[\cdot]$.

We will show that *u automatically solves an appropriate partial differential equation.* To see this, put $i(\tau) := I[u + \tau v]$, where v vanishes near ∂U. Since i has a minimum at $\tau = 0$, we can use the chain rule to compute

$$0 = i'(0) = \int_U (\nabla_v L \cdot \nabla v + L_z v) \, dx;$$

and so

$$0 = \int_U (-\mathrm{div}(\nabla_v L) + L_z) \, v \, dx,$$

in which L is evaluated at $(\nabla u, u, x)$. Here we write $\nabla_v L = (L_{v_1}, \dots, L_{v_n})$.

This integral identity is valid for all functions v vanishing on ∂U; whence follows the *Euler–Lagrange equation*

$$-\mathrm{div}(\nabla_v L(\nabla u, u, x)) + L_z(\nabla u, u, x) = 0. \quad (36)$$

Nonlinear Poisson equation. For example the Euler–Lagrange equation for

$$I[u] = \int_U \frac{1}{2}|\nabla u|^2 - F(u)\, dx,$$

is the *nonlinear Poisson equation*

$$-\Delta u = f(u) \quad (37)$$

where $f = F'$.

Minimal surfaces. The surface area of the graph of a function u is

$$I[u] = \int_U \left(1 + |\nabla u|^2\right)^{\frac{1}{2}}\, dx,$$

and the corresponding Euler–Lagrange equation is the *minimal surface equation*

$$\mathrm{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}}\right) = 0. \quad (38)$$

The expression on the left is (n times) the *mean curvature* of the surface; and consequently *a minimal surface has zero mean curvature.*

4.4 Maximum principles

The integral energy methods just discussed can for certain PDEs be augmented with pointwise maximum principle techniques. These are predicated upon the elementary observation that if the function u attains its maximum at an interior point x_0, then

$$u_{x_k}(x_0) = 0, \quad k = 1, \dots, n \quad (39)$$

and

$$\sum_{k,l=1}^{n} u_{x_k x_l}(x_0)\xi_k\xi_l \le 0, \quad \xi \in \mathbb{R}^n. \quad (40)$$

Linear elliptic equations. Such insights are essential for understanding the general *second-order linear elliptic equation*

$$Lu = 0, \quad (41)$$

where

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u.$$

We say L is *elliptic* provided the symmetric matrix $((a^{ij}(x)))$ is positive definite. In usual applications u represents the density of some quantity. The second-order term $\sum_{i,j=1}^n a^{ij}u_{x_i x_j}$ records *diffusion*, the first-order term $\sum_{i=1}^n b^i u_{x_i}$ represents *transport*, and the zeroth-order term cu describes the local *increase* or *depletion.*

We use the maximum principle to show for instance that if $c > 0$, then u *cannot attain a positive maximum at an interior point.* Indeed if u took on a positive maximum at some point x_0, then the first term of Lu at x_0 would be nonnegative (according to (40)), the next term would be zero (according to (39)) and the last would be positive. But this is a contradiction, since $Lu(x_0) = 0$.

Nonlinear elliptic equations. Maximum principle techniques apply also to many highly nonlinear equations, such as the *Hamilton–Jacobi–Bellman* equation

$$\max_{k=1,\dots,m}\{L^k u\} = 0. \quad (42)$$

This is an important equation in *stochastic optimization theory*, in which each elliptic operator L^k is the infinitesimal generator of a different stochastic process. We leave it to the reader to use the maximum principle to show that a solution of (42) cannot attain an interior maximum or minimum.

Related, but much more sophisticated maximum principle arguments can reveal many of the subtle properties of solutions to the linear elliptic equation (41) and the nonlinear equation (42).

4.5 Differential inequalities

Since solutions of PDEs depend on many variables, another useful trick is to design appropriate integral expressions over all but one of these variables, so that these expressions satisfy interesting differential inequalities in the remaining variable.

4.5.1 Dissipation estimates, gradient flows

For example, let $u = u(x,t)$ solve the *nonlinear gradient flow* equation

$$u_t - \mathrm{div}(\nabla L(\nabla u)) = 0 \tag{43}$$

in $\mathbb{R}^n \times (0,\infty)$. Put

$$e(t) := \frac{1}{2}\int_{\mathbb{R}^n} L(\nabla u)\,dx.$$

Then, assuming u goes to zero rapidly as $|x| \to \infty$, we have

$$\begin{aligned}\dot{e}(t) &= \int_{\mathbb{R}^n} \nabla L(\nabla u)\cdot \nabla u_t\,dx \\ &= -\int_{\mathbb{R}^n} (\mathrm{div}\nabla L(\nabla u))u_t\,dx = -\int_{\mathbb{R}^n} (u_t)^2\,dx \le 0.\end{aligned}$$

This is a dynamic *dissipation* inequality.

4.5.2 Entropy estimates

Related are *entropy* estimates for conservation laws. For this assume that $u^\epsilon = u^\epsilon(x,t)$ solves the *viscous conservation law*

$$u^\epsilon_t + F(u^\epsilon)_x = \epsilon u^\epsilon_{xx} \tag{44}$$

for $\epsilon > 0$. Suppose Φ is a convex function and put

$$e(t) := \int_{\mathbb{R}} \Phi(u^\epsilon)\,dx.$$

Then

$$\begin{aligned}\dot{e}(t) &= \int_{\mathbb{R}} \Phi' u^\epsilon_t\,dx = \int_{\mathbb{R}} \Phi'(-F_x + \epsilon u^\epsilon_{xx})\,dx \\ &= -\int_{\mathbb{R}} \left(\Psi(u^\epsilon)_x + \epsilon\Phi''(u^\epsilon_x)^2\right)\,dx \\ &= -\int_{\mathbb{R}} \epsilon\Phi''(u^\epsilon_x)^2\,dx \le 0,\end{aligned}$$

where Ψ satisfies $\Psi' = \Phi' F'$. What is important is that we have found not just one, but rather a large collection of dissipation inequalities, corresponding to each pair of *entropy/entropy flux* functions (Φ,Ψ).

Finding and utilizing *entropy/entropy flux* pairs for systems of conservation laws of the form (8) is a major challenge.

4.5.3 Monotonicity formulas

For *monotonicity formulas* we try to find interesting expressions to integrate over balls $B(0,r)$, with center say 0 and radius r. The hope is that these integral quantities will solve useful differential inequalities as functions of r.

As an example, consider the system

$$-\Delta \mathbf{u} = |D\mathbf{u}|^2\mathbf{u},\ |\mathbf{u}|^2 = 1. \tag{45}$$

for the unknown $\mathbf{u} = (u^1,\dots,u^m)$, where we write $|D\mathbf{u}|^2 = \sum_{i=1}^n\sum_{j=1}^m (u^j_{x_i})^2$.

A solution $\mathbf{u}$ is called a *harmonic map* into the unit sphere. It is a challenging exercise to derive from (45) the differential inequality

$$\begin{aligned}&\frac{d}{dr}\left(\frac{1}{r^{n-2}}\int_{B(0,r)} |D\mathbf{u}|^2\,dx\right) \\ &\qquad = \frac{2}{r^n}\int_{\partial B(0,r)} \textstyle\sum_{i,j,k} u^k_{x_i} x_i u^k_{x_j} x_j\,dS \ge 0,\end{aligned}$$

from which we deduce that

$$\frac{1}{r^{n-2}}\int_{B(0,r)} |D\mathbf{u}|^2\,dx \le \frac{1}{R^{n-2}}\int_{B(0,R)} |D\mathbf{u}|^2\,dx$$

if $0 < r < R$. This inequality is often useful, as it lets us deduce fine information at small scales r from that at larger scales R.

5 Theory and application

The foregoing listing of mathematical viewpoints and technical tricks provides at best a glimpse into the immensity of modern PDE theory, both pure and applied.

5.1 Well-posed problems

A common goal of most of these procedures is to understand a given PDE (plus appropriate boundary and/or initial conditions) as a *well-posed problem*, meaning that (a) the solution exists, (b) is unique, and (c) depends continuously on the given data for the problem. This is usually the beginning of wisdom, as well-posed problems provide the starting point for further theoretical inquiry, for numerical analysis and for construction of approximate solutions.

5.2 Generalized solutions

A central theoretical problem therefore is fashioning for any given PDE problem an appropriate notion of solution for which the problem is well-posed. For linear PDEs the concept of "distributional solutions" is usually the best, but for nonlinear problems there are many, including "viscosity solutions", "entropy solutions", "renormalized solutions", etc.

For example the unique *entropy solution* of the initial–value problem (2) for a scalar conservation law exists for all positive times, but may support lines of discontinuities across so-called *shock waves*. Similarly, the unique *viscosity solution* of the initial–value problem for the Hamilton–Jacobi equation (3) generally supports surfaces of discontinuity for its gradient. The explicit solution (30) for the porous medium equation is likewise not smooth everywhere and so needs suitable interpretation as a valid generalized solution.

The research literature teems with many such notions and some of the deepest insights in the field are uniqueness theorems for appropriate generalized solutions.

5.3 Learning more

This article, as promised, is a vast survey that actually explains precious little in any detail.

To learn more, interested students should definitely consult other articles in this book, as well as the following suggested reading. Markowich [7] is a nice introduction, with lots of pictures, and Strauss [10] is a very good undergraduate text, containing derivations of the various formulas cited here. Klainerman's survey article [6] is extensive and provides some different viewpoints. My graduate level textbook [3] carefully builds up much of the modern theory of PDEs, but is aimed at mathematically advanced students.

Further Reading

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