VARIOUS PROPERTIES OF SOLUTIONS OF THE INFINITY–LAPLACIAN EQUATION

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Abstract. We collect a number of technical assertions and related counterexamples about viscosity solutions of the infinity Laplacian PDE $-\Delta_\infty u = 0$ for $\Delta_\infty u := \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j}$.

1. Introduction.

This paper gathers together an assortment of new observations and counterexamples concerning viscosity solutions $u = u(x)$ of the "infinity-Laplacian" equation

$$-\Delta_\infty u = 0 \quad \text{in } U, \tag{1.1}$$

where U denotes an open subset of $\mathbb{R}^n$ and

$$\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j}. \tag{1.2}$$

Background and motivation. The PDE (1.1) is a sort of Euler-Lagrange equation for a basic calculus of variations problem in the sup-norm: see for instance the survey by Barron [B] and the references therein.

The fundamental variational principle, first identified by G. Aronsson, is this. We are given a continuous function $g : \partial U \to \mathbb{R}$ and are asked to extend g to a function $u : \bar{U} \to \mathbb{R}$ so that

$$\begin{cases} \text{for each bounded subset } V \text{ of } U \text{ and each function } v \in C(\overline{V}), \\ u = v \text{ on } \partial V \quad \text{implies} \quad \text{ess-sup}_V\, |Du| \leq \text{ess-sup}_V\, |Dv|. \end{cases} \tag{1.3}$$

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We say that u is an "absolutely minimizing Lipschitz extension" of the boundary values g.

A central question has been to understand this minimization problem and to develop appropriate PDE methods. It is a central discovery of G. Aronsson that the Euler-Lagrange equation of the (1.3), suitably interpreted, is a highly nonlinear and highly degenerate PDE (1.1). It has been a major challenge to understand the connections between analytic properties of solutions to the boundary–value problem

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } U \\ u = g & \text{in } \partial U; \end{cases} \tag{1.4}$$

and the variational problem associated with (1.3).

R. Jensen [J] proved the first basic theorems asserting that the Dirichlet problem (1.4) is well-posed, He showed that if U is a bounded, open, connected subset of $\mathbb{R}$ and $g \in C(\partial U)$, then (1.4) has a *unique* viscosity solution $u \in C(\overline{U})$ satisfying $u = g$ on ∂U.

More recently, Crandall, Evans and Gariepy [C-E-G] have focussed attention upon the "comparison with cones" property of solutions. They introduced quantities

$$S^+(x,r) := \max_{y \in B(x,r)} \frac{u(y) - u(x)}{r} \geq 0$$

and

$$S^-(x,r) := \min_{y \in B(x,r)} \frac{u(y) - u(x)}{r} \leq 0,$$

defined if $B(x,r) \subset U$, and showed that the mappings $r \mapsto S^+(x,r), S^-(x,r)$ are respectively monotone increasing and decreasing. Thus the limits

$$S(x) := \lim_{r \to 0} S^+(x,r) = -\lim_{r \to 0} S^-(x,r) \tag{1.5}$$

exist and, as shown in [C-E-G], are equal. If u is differentiable at x, then of course $S(x) = |Du(x)|$.

These observations raise the fundamental regularity question as to whether viscosity solutions of (1.4) are everywhere differentiable or even C^1. This still unsolved problem is the motivation for much of the new mathematics in our paper. (Very recently O. Savin [S] has shown in an important paper that viscosity solutions of (1.1) in two dimensions are in fact continuously differentiable.)

Outline of this paper. Our paper is a collection of interesting, but only weakly interconnected, topics.

In Section 2 we introduce a measure of the "flatness" of the graph of u on a ball, namely the sup norm of the error when we approximate u by a plane. We then demonstrate by an example that, unfortunately, this flatness does not decrease by a factor less than one if we pass to a smaller, concentric ball. This failure of the flatness to improve is somewhat unexpected, since essentially all conventional proofs of C^1 regularity (or partial regularity) depend on such an assertion.

Section 3 provides some techniques of "approximation by cones" from above and below. We show that if u is flat on some ball, then we can in better approximate u by cones on a smaller ball. We think this technical statement is somewhat interesting, and pretty encouraging, but so far we know of no substantial application.

Section 4 demonstrates that an infinity harmonic function as being a weak solution of a divergence structure PDE of the form

$$\operatorname{div}(\sigma Du) = 0, \tag{1.6}$$

for a probability measure σ, the support of which lies within the set where $|Du|$ attains its maximum.

We look more closely in Section 5 at the approximate PDE introduced in §4, and provide some Jacobian estimates for the related "characteristic" flows. These in part explain some puzzling features of certain specific solutions of $-\Delta_\infty = 0$.

Finally, in Section 6 we deduce some behaviors of solutions under symmetry assumptions and for certain boundary conditions.

As the foregoing listing of topics makes clear, this real theme of this paper is developing and advertising some novel viewpoints for the infinity Laplacian equation. We hope our gathering these diverse ideas together will encouage further progress on the fascinating, if frustrating, PDE.

2. Failure of flatness decay.

2.1 Flatness decay. Suppose $x^0 \in U$ and that the ball $B(x^0, R)$ lies in U. Then for a radius $0 < r \leq R$, we can measure how well u restricted to $B(x^0, r)$ can be approximated by a linear function in terms of the *flatness*

$$E(x^0, r) := \min_{a \in \mathbb{R}, b \in \mathbb{R}^n} \max_{B(x^0,r)} \frac{|u - a - b \cdot (x - x^0)|}{r}. \tag{2.1}$$

The *flatness decay conjecture* asserts that

$$\begin{cases} \text{there exist constants } \varepsilon_0 > 0,\ 0 < \tau, \eta < 1, \text{ such that} \\ \text{for any solution } u \text{ and any ball } B(x^0, r) \subset U, \\ E(x^0, r) < \varepsilon_0 \text{ implies } E(x^0, \tau r) \leq \eta E(x^0, r). \end{cases} \tag{2.2}$$

This assertion, if valid, would imply that $E(x, r) \leq C r^\gamma$ for some $\gamma > 0$, all radii $0 < r \leq R$ and all points x sufficiently close to x^0. And then we could deduce that u is $C^{1,\gamma}$ near x^0.

Most regularity (or partial regularity) assertions for elliptic PDE follow from some variant of an flatness decay assertion like (2.2). So it is unfortunate that the conjecture is false for solutions of the infinity-Laplacian PDE (although, as O. Savin [S] has recently shown, a solution in two dimensions is in fact C^1).

2.2 A counterexample. We will take $n = 2$, $x^0 = 0$, $R = 1$, and build a collection $\{u^\lambda\}_{0<\lambda\leq\lambda_0}$ of $C^{1,1}$ functions solving

$$\Delta_\infty u^\lambda = 0 \quad \text{in } U = B(0,1), \tag{2.3}$$

such that

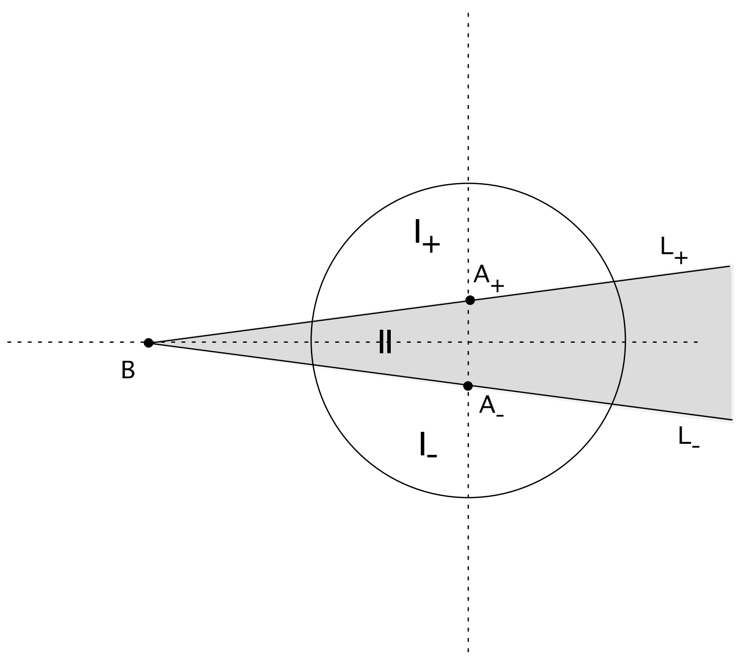
$$E^\lambda(0,1) \to 0; \tag{2.4}$$

but for all $0 < \tau, \eta < 1$

$$E^\lambda(0,\tau) > \eta E^\lambda(0,1) \quad \text{if } \lambda \text{ is small enough.} \tag{2.5}$$

Here $E^\lambda(0,r)$ denotes the flatness of u^λ on $B(0,r)$:

$$E^\lambda(0,r) := \min_{a,b} \max_{B(0,r)} \frac{|u^\lambda(x) - a - b\cdot x|}{r}.$$



Construction of counterexample

Definition of $\mathbf{u}^\lambda$. Referring to the illustration, consider the line L_+ which makes an angle $\lambda > 0$ when it crosses the x_1-axis, at the point $B = (-2, 0)$. L_- is its reflection across the x_1-axis.

Let I_+ denote that part of the disk $U = B(0,1)$ lying above L_+, II that part lying between L_+ and L_-, and I_- that part lying below L_-. The marked point A_+ is $(0, 2\tan\lambda)$ and A_- is $(0, -2\tan\lambda)$.

Define

$$\phi(x) := |x - B| - 2 = ((x_1 + 2)^2 + x_2^2)^{1/2} - 2. \tag{2.6}$$

Then $\phi \in C^\infty(B(0,1))$, $|D\phi| = 1$. Finally, put

$$u^\lambda(x) := \begin{cases} \phi(A_+) + D\phi(A_+) \cdot (x - A_+) & \text{if } x \in I_+ \\ \phi(x) & \text{if } x \in II \\ \phi(A_-) + D\phi(A_-) \cdot (x - A_-) & \text{if } x \in I_-. \end{cases} \tag{2.7}$$

The graph of u^λ therefore comprises a piece of a cone, glued to two planes along the lines $L_\pm$.

Lemma 2.1. *The function u^λ is a viscosity solution of*

$$-\Delta_\infty u^\lambda = 0 \quad \textit{in } U.$$

Proof. Let ϕ be a smooth function and assume $u^\lambda - \phi$ has a maximum (minimum) at a point $x_0 \in U$. We must prove

$$-\Delta_\infty \phi(x_0) \leq 0 \quad (\geq 0).$$

This is clear if u^λ is C^2 in a neighborhood of x_0, since $|Du^\lambda| \equiv 1$.

Suppose instead $x_0 \in L_\pm$ and $u^\lambda(x_0) = \phi(x_0)$. Since u^λ is everywhere C^1, we have $Du^\lambda(x_0) = D\phi(x_0) =: \xi$. The vector ξ points along the line $L_\pm$ and we must show

$$\phi_{\xi\xi}(x_0) \geq 0 \quad (\leq 0).$$

But these inequalities hold since $\phi \geq u^\lambda$ ($\phi \leq u^\lambda$) along $L_\pm$, $u^\lambda(x_0) = \phi(x_0)$, and u^λ is linear on $L_\pm$. □

Lemma 2.2. (i) *We have*

$$\lim_{\lambda \to 0} \frac{u^\lambda(x) - x_1}{\lambda} = |x_2|, \qquad \textit{uniformly for } x \in B(0,1), \tag{2.8}$$

and

$$\max_{B(0,1)} \left| u^\lambda(x) - \frac{\lambda}{2} - x_1 \right| = \frac{\lambda}{2}(1 + o(1)) \quad \text{as } \lambda \to 0. \tag{2.9}$$

(ii) *Furthermore, for each* $0 < \tau < 1$,

$$\lim_{\lambda \to 0} \min_{a \in \mathbb{R}, b \in \mathbb{R}^2} \max_{B(0,\tau)} \frac{|u^\lambda(x) - a - b \cdot x|}{\lambda \tau} \geq \frac{1}{2} \tag{2.10}$$

and

$$\lim_{\lambda \to 0} \max_{B(0,\tau)} \frac{\left| u^\lambda(x) - \frac{1}{2}\lambda\tau - x_1 \right|}{\lambda \tau} = \frac{1}{2}. \tag{2.11}$$

Proof. 1. We note that in the ball $B(0,1)$ we have

$$D\phi(x) = \frac{1}{|x - B|}(x_1 + 2, x_2)$$

and

$$D\phi(A_\pm) = (1 + \sin^2 \lambda)^{-1/2}(1, \pm \sin \lambda).$$

Write

$$v^\lambda(x) := \frac{u^\lambda(x) - x_1}{\lambda}.$$

If $x \in I_\pm$, then

$$Dv^\lambda = \frac{D\phi(A_+) - (1,0)}{\lambda} = \left(\frac{(1 + \sin^2 \lambda)^{-1/2} - 1}{\lambda}, \pm (1 + \sin^2 \lambda)^{1/2} \frac{\sin \lambda}{\lambda} \right);$$

and so $|Dv^\lambda|$ is bounded in $I_+ \cup I_-$, uniformly as $\lambda \to 0$. If $x \in II$, then

$$Dv^\lambda = \frac{D\phi(x) - (1,0)}{\lambda} = \frac{1}{|x - B|} \left(\frac{x_1 + 2 - |x - B|}{\lambda}, \frac{x_2}{\lambda} \right).$$

Observe $|x_2|$ is $O(\lambda)$ in II, and

$$(x_1 + 2) - |x - B| = (x_1 + 2) - ((x_1 + 2)^2 + x_2^2)^{1/2} \quad \text{is } O(\lambda^2).$$

Thus $|Dv^\lambda|$ is bounded as well in II, uniformly as $\lambda \to 0$.

2. If $x \in I_\pm$, we have

$$\begin{aligned} v^\lambda(x) &= \frac{1}{\lambda}[\phi(A_\pm) + D\phi(A_\pm) \cdot (x - A_\pm) - x_1] \\ &= \frac{1}{\lambda}[(4 + 4\sin^2 \lambda)^{1/2} - 2 + (1 + \sin^2 \lambda)^{-1/2}(x_1 \pm \sin \lambda x_2 - 2\sin^2 \lambda) - x_1]. \end{aligned}$$

Consequently

$$\lim_{\lambda\to 0} v^\lambda(x) = \pm x_2 = |x_2|.$$

Since the functions $\{v^\lambda\}$ are uniformly Lipschitz continuous and $|\,II\,| \to 0$ as $\lambda \to 0$, this implies (2.8). Since therefore

$$u^\lambda(x) - x_1 = \lambda|x_2| + o(\lambda),$$

we have

$$\begin{aligned}\max_{B(0,1)} \left| u^\lambda(x) - \frac{\lambda}{2} - x_1 \right| &= \max_{B(0,1)} \left| \lambda \left(|x_2| - \frac{1}{2} \right) \right| + o(\lambda) \\ &= \frac{\lambda}{2}(1 + o(1)).\end{aligned}$$

This proves (2.9).

3. Likewise

$$\begin{aligned}\max_{B(0,\tau)} \frac{\left| u^\lambda(x) - \frac{\lambda\tau}{2} - x_1 \right|}{\tau} &= \max_{B(0,\tau)} \frac{\left| \lambda \left(|x_2| - \frac{\tau}{2} \right) \right|}{\tau} + o(\lambda) \\ &= \frac{\lambda}{2}(1 + o(1)),\end{aligned}$$

and this is (2.11).

4. We must lastly confirm (2.10), which says we cannot somehow improve the linear approximation by replacing x_1 by $a + b \cdot x$. Suppose instead

$$\max_{B(0,\tau)} \frac{|u^\lambda - a - b \cdot x|}{\tau\lambda} \le \eta < \frac{1}{2} \tag{2.12}$$

for sufficiently small λ. Then for all $x \in B(0,\tau)$,

$$|x_1 + \lambda \mid x_2| - a - b_1 x_1 - b_2 x_2| \le \eta\lambda\tau + o(\lambda) \le \gamma\lambda\tau \tag{2.13}$$

for some $\eta < \gamma < \frac{1}{2}$.

Assume first $b_2 > 0$. Take $x_1 = 0$, $x_2 = -\tau$ in (2.13):

$$|(\lambda + b_2)\tau - a| \le \gamma\lambda\tau.$$

Thus

$$a \ge (\lambda + b_2 - \gamma\lambda)\tau.$$

Now take $x_1 = x_2 = 0$ in (2.13), to find $|a| \le \gamma\lambda\tau$. Thus

$$(\lambda + b_2 - \gamma\lambda)\tau \le \gamma\lambda\tau;$$

and so we derive the contradiction

$$0 < b_2 < (2\gamma - 1)\lambda < 0,$$

as $\gamma < \frac{1}{2}$. We similarly deduce that $b_2 < 0$ is impossible, and so $b_2 = 0$.

Thus (2.12) now reads

$$|x_1 + \lambda|x_2| - a - b_1 x_1| \le \gamma\lambda\tau$$

for $\gamma < \frac{1}{2}$, for $c = 1 - b_1$. Let $x_1 = x_2 = 0$, to learn

$$|a| \le \gamma\lambda\tau.$$

Let $x_1 = 0$, $x_2 = \tau$, to find also that

$$|\lambda\tau - a| \le \gamma\lambda\tau.$$

Thus

$$\lambda\tau \le 2\lambda\gamma\tau,$$

which is impossible since $\gamma < \frac{1}{2}$.

We have proved that if λ is small enough there do not exist $a, \eta \in \mathbb{R}$, $b \in \mathbb{R}$ such that (2.12) holds. This establishes (2.10). □

According to (2.9), $E^\lambda(0,1) \le \frac{\lambda}{2}(1 + o(1))$ (and in fact we have equality). Owing to (2.10) and (2.11),

$$E^\lambda(0,\tau) \ge \frac{\lambda}{2}(1 + o(1)) \quad \text{for each } 0 < \tau < 1.$$

Hence (2.2), (2.4) are valid: the flatness decay conjecture fails.

2.3 Comments on the blow-up method. Flatness decay type assertions are often proved by a "blow-up" procedure, which for the case at hand would proceed as follows: To (try to) prove (2.2), suppose the contrary and (try to) derive a contradiction. So suppose for given $0 < \tau$, $\eta < 1$ that there exist balls $B(x_k, r_k) \subset U$ for which

$$E(x_k, r_k) =: \lambda_k \to 0,$$

but

$$E(x_k, \tau r_k) > \eta\lambda_k.$$

Here

$$E(x_k, r_k) = \max_{B(x_k, r_k)} \frac{|u - a^k - b_i^k(x - x_k)|}{r_k}.$$

We rescale, by writing for $x \in B(0,1)$

$$v^k(x) := \frac{u(x_k + r_k x) - a^k - r_k b^k \cdot x}{\lambda_k r_k}.$$

Then

$$E^k(0,1) = 1, \text{ but } E^k(0,\tau) > \eta, \tag{2.14}$$

E^k denoting the flatness for v^k.

Assume now $v^k \to v$ uniformly as $k \to \infty$, and find a PDE the "blow-up limit" v satisfies. To do so we may assume as well $b^k \to b$. If $b \neq 0$, we may without loss assume $b = e_1 = (1, \dots, 0, 0)$.

Proceeding formally, we calculate that

$$v^k_{x_i} = \frac{u_{x_i}(x_k + r_k x) - b^k_i}{\lambda_k}, \; v^k_{x_i x_j} = \frac{r_k}{\lambda_k} u_{x_i x_j}.$$

So

$$0 = \Delta_\infty u = u_{x_k} u_{x_j} u_{x_i x_j} = \frac{\lambda_k}{r_k}(b^k_i + \lambda_k v^k_{x_i})(b^k_j + \lambda_k v^k_{x_j}) v^k_{x_i x_j}.$$

Cancelling the first term and letting $k \to \infty$, $\lambda^k \to 0$, $b^k_i \to \delta_{in}$, we deduce that v solves the PDE

$$v_{x_1 x_1} = 0. \tag{2.15}$$

The foregoing calculations can in fact be made rigorous in the sense of viscosity solutions.

The general solution of (2.15) has the form

$$v(x) = a(x_2, \dots, x_n) + b(x_2, \dots, x_n) x_1. \tag{2.16}$$

The point is that the highly degenerate PDE (2.15) is, under our normalizations, the blow-up limit of the infinity Laplacian. But since (2.15) only implies (2.16), with no information at all about the functions $a(\cdot)$ and $b(\cdot)$. If $a(\cdot)$ and $b(\cdot)$ were known to be $C^{1,\alpha}$, we could in fact derive a contradiction to (2.14) and thereby prove flatness decay. Our counterexample in §2.2 shows all this is impossible, since we obtain

$$a(x_2) = |x_2|, \quad b \equiv 0.$$

Our function $a(\cdot)$ is Lipschitz continuous, and not C^1. The moral seems to be that the linearization (2.15) of infinity Laplacian is too degenerate to be useful.

Remark. Savin's recent paper [S] shows, without employing any sort of flatness decay estimate, that viscosity solutions of (1.1) in two dimensions are C^1 . The key here problem is to understand how the flatness of the solution on, say, the ball $B(x,R)$ controls the flatness on the much smaller ball $B(x,r)$, for $0 < r << R$. For this Savin makes use of a topological argument, valid in two dimensions, showing that an approximate tangent plane must cut the graph into various disconnected components. □

3. Approximation by cones.

In spite of the counterexample built in §2, we show next that if the flatness $E(x^0, r)$ is small, then u can be "well approximated from above and below by cones" on a smaller ball $B(x^0, \tau r)$. This assertion in effect shows that some sort of improvement in flatness occurs if we pass to a smaller ball, although we do not know any way to exploit this observation effectively.

Without loss we may assume $x^0 = 0$, $r = 1$, $u(0) = 0$. Fix $0 < \tau \leq \frac{1}{4}$.

Theorem 3.1. *Define*

$$\lambda := \max_{B(0,1)} |u - x_n|.$$

(i) *There exist points* $x^{\pm} \in \{u = \pm\frac{1}{2}\}$ *such that*

$$C^-(x) \leq u(x) \leq C^+(x) \quad (x \in B(0, \tau)) \tag{3.1}$$

for certain cones

$$\begin{cases} C^+(x) = a^+ + b^+|x - x^-|, \\ C^-(x) = a^- - b^-|x - x^+|. \end{cases} \tag{3.2}$$

(ii) *Let* $L = [x^-, x^+]$ *denote the line segment from* x^- *to* x^+*. We can select* $x^{\pm}$, $a^{\pm}$, $b^{\pm}$ *as above so that in addition*

$$L \text{ passes through } B(0, \tau), \tag{3.3}$$

$$|C^+(x) - C^-(x)| \leq \frac{C\lambda^{3/2}}{\tau} \quad (x \in L), \tag{3.4}$$

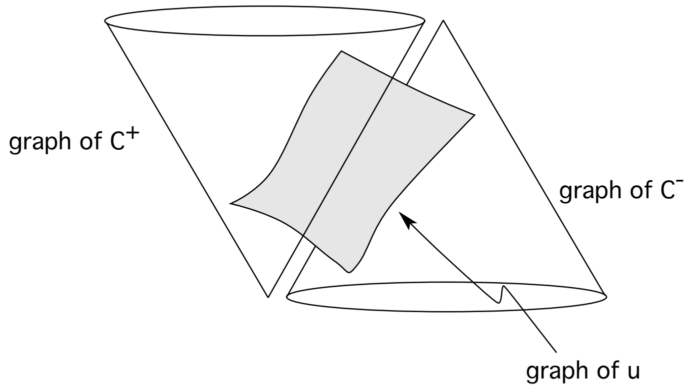
and

$$|b^+ - b^-| \leq \frac{C\lambda^{3/2}}{\tau}. \tag{3.5}$$

We are saying that the graph of u on $B(0, \tau)$ lies between the graphs of the upward opening cone C^+ and the downward opening cone C^-, as illustrated. Furthermore, these cones are close along the line segment L, being separated by a factor $\frac{C\lambda^{3/2}}{\tau} \ll \lambda$ for $\lambda \ll 1$.

Proof. 1. Define $\Lambda^{\pm} = \{x \in B(0, 1) \mid u = \pm\frac{1}{2}\}$. We will write $x = (x', x_n)$ for $x' = (x_1, \ldots, x_{n-1})$. Select $x_0 \in \Lambda^-$ with $x_0' = 0$. Then take $x_1 \in \Lambda^+$ such that $|x_0 - x_1| = \text{dist}(x_0, \Lambda^+)$. Continue, inductively finding for even k

$$x_k \in \Lambda^- \quad \text{such that } |x_k - x_{k-1}| = \text{dist}(x_{k-1}, \Lambda^-), \tag{3.6}$$



Approximation by cones from above and below

and for odd k,

(3.7) $$x_k \in \Lambda^+ \quad \text{such that } |x_k - x_{k-1}| = \operatorname{dist}(x_{k-1}, \Lambda^+).$$

2. Write

$$r_k := |x_k - x_{k-1}|$$

We observe first that

(3.8) $$r_{k+1} \le r_k \quad \text{for } k = 1, 2, \dots$$

To see this, we note that for even k,

$$S^+(x_k, r_k) = \frac{u(x_{k+1}) - u(x_k)}{r_k} = \frac{1}{r_k},$$

since $u(x_{k+1}) = \frac{1}{2}$, $u(x_k) = -\frac{1}{2}$. Also

$$S^-(x_{k+1}, r_{k+1}) = \frac{u(x_{k+2}) - u(x_{k+1})}{r_{k+1}} = -\frac{1}{r_{k+1}}.$$

But $S^+(x_k, r_k) \le -S^-(x_{k+1}, r_{k+1})$, and so (3.8) follows. Similar reasoning applies if k is odd.

3. Next, notice that

(3.9) $$x \in \Lambda^\pm \quad \text{implies} \quad \left| x_n \mp \frac{1}{2} \right| \le \lambda;$$

this means the level sets $\Lambda^\pm$ have "thickness" at most λ in the x_n direction. Consequently

(3.10) $$r_k \le 1 + C\lambda, \ r_k \ge 1 - 2\lambda \quad \text{for } k = 1, 2, \dots$$

Therefore

$$r_k \in [1-2\lambda, 1+C\lambda] \quad (k=1,2,\dots).$$

This and (3.8) implies that if we take $k=1,2,\dots,N-1$ (N to be selected), there must exist two consecutive radii such that

$$0 \le r_k - r_{k+1} \le \frac{C\lambda}{N}. \tag{3.11}$$

4. We next estimate how many times N we can carry out our "zig-zag" construction and be certain the line segment $[x_k, x_{k+1}]$ intersects the ball $B(0,\tau)$.

Assume, say, k is odd; so that $x_k = (x'_k, x_{k,n})$ lies on Λ^+. Let

$$y_{k+1} = (x'_k, y_{k+1,n})$$

be a point on Λ^- with the same first $(n-1)$-coordinates as x_k. Let $x_{k+1} = (x'_{k+1}, x_{k+1,n})$ denote, as above, a point on Λ^- closest to x_k.

Then

$$|x_{k+1} - x_k| \le |y_{k+1} - x_k|.$$

Squaring and expanding, we deduce

$$|x'_{k+1} - x'_k|^2 + (x_{k+1,n} - x_{k,n})^2 \le (y_{k+1,n} - x_{k,n})^2;$$

and so

$$\begin{aligned} |x'_{k+1} - x'_k|^2 &\le y^2_{k+1,n} - 2y_{k+1,n}x_{k,n} \\ &\quad - x^2_{k+1,n} + 2x_{k+1,n}x_{x,n} \\ &\le C|y_{k+1,n} - x_{k+1,n}| \\ &\le C\lambda, \end{aligned}$$

the last inequality holding according to (3.9). Hence

$$|x'_{k+1} - x'_k| \le C\lambda^{1/2}. \tag{3.12}$$

Since $x'_0 = 0$, we have

$$|x'_k| \le Ck\lambda^{1/2}.$$

Hence the line segment $[x_k, x_{k+1}]$ will intersect the ball $B(0,\tau)$ provided

$$k \le C\tau\lambda^{-1/2}.$$

We can consequently set $N = C\tau\lambda^{-1/2}$ in (3.11): There exist radii such that

$$0 \le r_k - r_{k+1} \le \frac{C\lambda^{3/2}}{\tau}. \tag{3.13}$$

5. We may suppose hereafter that k is even. Write

$$\begin{cases} x^+ = x_{k-1},\ x^- = x_k, \\ a^+ = -\frac{1}{2},\ b^+ = \frac{1}{r_k}, \\ a^- = \frac{1}{2},\ b^- = \frac{1}{r_{k+1}} \end{cases}$$

and define $C^{\pm}(\cdot)$ by (3.2). Note that

$$C^+(x^+) = -\frac{1}{2} + \frac{1}{r_k}|x^+ - x^-| = \frac{1}{2} = C^-(x^+);$$

and

$$|b^+ - b^-| = \frac{r_k - r_{k+1}}{r_k r_{k+1}} \leq \frac{C\lambda^{3/2}}{\tau},$$

by (3.13). This gives (3.5), and (3.1) follows by comparison with cones. Lastly we note that since C^+ and C^- agree at x^+, estimate (3.5) implies (3.4). □

4. Concentration measures and a divergence-structure PDE.

In this section we reexamine the singular variational principle giving rise to the operator Δ_∞ and extract a related divergence structure equation. These assertions are motivated by the heuristic paper [E2], which notes the quite similar mathematical structures for the infinity Laplacian problem, Monge-Kantorovich mass transfer problems and weak KAM theory.

4.1 A minimization problem, limits as $k \to \infty$. For fixed $k > 0$, let u^k be the unique minimizer of the functional

(4.1)
$$I[v] := \int_U e^{\frac{k}{2}|Dv|^2} dx$$

among Lipschitz continuous mappings with $v = g$ on ∂U. Then $u^k \in C(\bar{U}) \cap C^\infty(U)$ and

(4.2)
$$\begin{cases} \operatorname{div}(e^{\frac{k}{2}|Du^k|^2} Du^k) = 0 & \text{in } U \\ u^k = g & \text{on } \partial U. \end{cases}$$

In particular,

(4.3)
$$-u^k_{x_i} u^k_{x_j} u^k_{x_i x_j} - \frac{1}{k}\Delta u^k = 0 \quad \text{in } U.$$

We define also

(4.4)
$$L_k^2 := \frac{2}{k} \log\left(⨍_U e^{\frac{k}{2}|Du^k|^2} dx \right),$$

the slash through the integral denoting the average. Hence if we set

$$\sigma^k := \frac{e^{\frac{k}{2}(|Du^k|^2 - L_k^2)}}{|U|}, \tag{4.5}$$

we have

$$\sigma^k > 0, \ \int_U \sigma^k dx = 1, \tag{4.6}$$

and

$$\operatorname{div}(\sigma^k Du^k) = 0 \quad \text{ in } U. \tag{4.7}$$

Finally, write

$$L := \sup \left\{ \frac{|g(x) - g(y)|}{\operatorname{dist}_U(x,y)} \ \middle| \ x, y \in \bar{U}, x \neq y \right\}, \tag{4.8}$$

where $d_U(x,y)$ denotes the distance from x to y within U. (If U is convex, $d_U(x,y) = |x-y|$.)

Our goal in this section will be passing to limits in the divergence structure PDE (4.7) as $k \to \infty$.

Lemma 4.1. (i) *We have*

$$L_k \leq L. \tag{4.9}$$

(ii) *Furthermore for each $n < p < \infty$,*

$$\sup_k \|u^k\|_{W^{1,p}(U)} < \infty. \tag{4.10}$$

Proof. 1. Let u solve

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } U \\ \qquad u = g & \text{in } \partial U; \end{cases}$$

so that

$$\operatorname*{ess\,sup}_U |Du| = L.$$

Then

$$⨍_U e^{\frac{k}{2}|Du_k|^2} dx \leq ⨍_U e^{\frac{k}{2}|Du|^2} dx \leq e^{\frac{k}{2}L^2}.$$

Hence (4.4) implies $L_k^2 \leq L^2$.

2. Now fix $n < p < \infty$ and take $k \geq p$. Then since $e^x \geq x$ for $x \geq 0$, we have

$$(4.11) \qquad \begin{aligned} \left(\fint_U \left(\frac{|Du^k|^2}{2} \right)^p dx \right)^{1/p} &\leq \left(\fint_U e^{\frac{p}{2}|Du^k|^2} dx \right)^{1/p} \\ &\leq \left(\fint_U e^{\frac{k}{2}|Du^k|^2} dx \right)^{1/k} \\ &= e^{\frac{1}{2}L_k^2} \end{aligned}$$

Hence

$$\left(\fint_U |Du^k|^{2p} dx \right)^{\frac{1}{2p}} \leq 2e^{\frac{L^2}{4}} \quad \text{for } k \geq p.$$

□

Limits as k→∞. In view of (4.10), $\{u^k\}_{k=1}^{\infty}$ is bounded and uniformly equicontinuous. Hence for some subsequence $k_j \to \infty$, we have $u^{k_j} \to u$ uniformly on $\bar{U}$. In view of (4.3), u is a viscosity solution of

$$(4.12) \qquad \begin{cases} -\Delta_\infty u = 0 & \text{in } U \\ u = g & \text{in } \partial U. \end{cases}$$

By uniqueness, in fact the full sequence converges:

$$(4.13) \qquad u^k \to u \quad \text{uniformly on } \bar{U}.$$

Also, according to (4.6) we may assume

$$(4.14) \qquad \sigma^{k_j} \rightharpoonup \sigma \quad \text{weakly as measures on } \bar{U}$$

where σ is a Radon measure on $\bar{U}$, normalized so that

$$\sigma(\bar{U}) = 1.$$

4.2 A divergence structure equation Our intention next is to pass to limits in the PDE (4.7) as $k \to \infty$, showing thereby that "div$(\sigma Du) = 0$" in an appropriate weak sense. We hereafter write

$$d\sigma^k := \sigma^k dx = \frac{e^{\frac{k}{2}(|Du^k|^2 - L_k^2)}}{|U|} dx.$$

Lemma 4.2. (i) *We have*

$$\lim_{k\to\infty} L_k = L \quad \textit{and} \quad \lim_{k\to\infty}\int_U |Du^k|^2\,d\sigma^k = L^2. \tag{4.15}$$

(ii) *If* $v \in C^{0,1}(\bar{U})$ *satisfies*

$$v = g \text{ on } \partial U, \quad |Dv| \le L \text{ a.e.,} \tag{4.16}$$

then

$$\lim_{k\to\infty}\int_U |Du^k - Dv|^2\,d\sigma^k = 0. \tag{4.17}$$

Proof. 1. As before take $k \ge p$. Then

$$\left(\fint_U e^{\frac{p}{2}|Du^k|^2}dx\right)^{1/p} \le \left(\fint_U e^{\frac{k}{2}|Du^k|^2}dx\right)^{1/k} = e^{\frac{L_k^2}{2}}.$$

Assume $\liminf_{k\to\infty} L_k =: M \le L$. Then there exists a subsequence $k_j \to \infty$ such that

$$L_{k_j} \to M.$$

Also $u^{k_j} \to u$ uniformly on $\bar{U}$, where u is the unique viscosity solution of (4.12). By lower semicontinuity of the L^p-norm, we have

$$\left(\fint_U e^{\frac{p}{2}|Du|^2}dx\right)^{1/p} \le e^{\frac{M^2}{2}}$$

for all p. Therefore

$$\operatorname*{ess-sup}_U e^{\frac{|Du|^2}{2}} \le e^{\frac{M^2}{2}}.$$

Since ess-sup $|Du| = L \ge M$, this is a contradiction unless $M = L$.

2. We note next that for each $\eta > 0$,

$$\begin{aligned}\int_U |Du^k|^2\,d\sigma^k &= \int_{\{|Du^k|^2 \ge L_k^2-\eta\}} |Du^k|^2\,d\sigma^k + \int_{\{|Du^k|^2 < L_k^2-\eta\}} |Du^k|^2\,d\sigma^k \\ &=: A + B.\end{aligned}$$

Now

$$B \le (L_k^2 - \eta)\fint_U e^{-\frac{k\eta}{2}}\,dx \to 0$$

and

$$A \ge (L_k^2-\eta)\sigma^k(\{|Du^k|^2 \ge L_k^2-\eta\}) = (L_k^2-\eta)(1-\sigma^k(\{|Du^k|^2 \le L_k^2-\eta\}).$$

Thus for each $\eta > 0$,

$$\liminf_{k\to\infty} \int_U |Du^k|^2 \, d\sigma^k \geq L^2 - \eta. \tag{4.18}$$

2. In view of (4.7), we have

$$\int_U Du^k \cdot (Du^k - Dv) \, d\sigma^k = 0$$

if $v = g$ on ∂U. Consequently,

$$\begin{aligned} \int_U |Du^k|^2 \, d\sigma^k &= \int_U Du^k \cdot Dv \;\; d\sigma^k \\ &= \int_U \left(\frac{1}{2}|Du^k|^2 + \frac{1}{2}|Dv|^2 - \frac{1}{2}|Du^k - Dv|^2 \right) d\sigma^k; \end{aligned}$$

and so

$$\int_U |Du^k - Dv|^2 \, d\sigma^k + \int_U |Du^k|^2 \, d\sigma^k = \int_U |Dv|^2 d\sigma^k. \tag{4.19}$$

If we take $v = u$, the solution of (4.12), then $|Dv| = |Du| \leq L$ a.e. and so

$$\limsup_{k\to\infty} \int_U |Du^k|^2 \, d\sigma^k \leq L^2.$$

This and (4.18) prove the second limit asserted in (4.15). Furthermore, if we take any v satisfying (4.16) in (4.19), we have

$$\limsup_{k\to\infty} \int_U |Du^k - Dv|^2 \, d\sigma^k \leq L^2 - \lim_{k\to\infty} \int_U |Du^k|^2 \, d\sigma^k = 0.$$

□

Now define for $x \in U$ the *upper* and *lower extensions*

$$u^-(x) := \min_{y\in\partial U} \{g(y) + L d_U(x, y)\}$$

and

$$u^+(x) := \max_{y\in\partial U} \{g(y) - L d_U(x, y)\}.$$

(We follow here the notation of A. Fathi for weak KAM theory: see [F].) Then $u^- = u^+ = g$ on ∂U, and

$$u^+ \leq u \leq u^- \quad \text{in } \bar{U}. \tag{4.20}$$

Introduce also the *touching set*

$$T := \{x \in U \mid u^-(x) = u^+(x)\}.$$

Theorem 4.3. *We have*

$$\operatorname{spt}\sigma \cap U \subseteq T; \tag{4.21}$$

and consequently $Du(x)$ exists for each point $x \in \operatorname{spt}(\sigma)$.

Proof. 1. We first *claim* that

$$|Du| < L \text{ a.e. on } U - T. \tag{4.22}$$

To see this, suppose first that

$$u(x_0) < u^-(x_0) \tag{4.23}$$

for some point $x_0 \in U$, at which $Du(x_0)$ exists. Define the cone function

$$c(x) := u(x_0) - M d_U(x, x_0),$$

for $M > 0$ selected later. Clearly $c(x_0) = u(x_0)$. Suppose now $y \in \partial U$. Then

$$\begin{aligned}
c(y) &= u(x_0) - M d_U(y, x_0) \\
&= u(x_0) - u^-(x_0) + u^-(x_0) - M d_U(y, x_0) \\
&\leq -(u^-(x_0) - u(x_0)) + g(y) + (L - M) d_U(y, x_0) \\
&= g(y) + (L - M) d_U(x_0, \partial U) - (u^-(x_0) - u(x_0)) \\
&= g(y)
\end{aligned}$$

for

$$M := L - \frac{u^-(x_0) - u(x_0)}{d_U(x_0, \partial U)}. \tag{4.24}$$

By comparison, we have $u(x) \geq c(x)$ for all $x \in U$, and consequently

$$|Du(x_0)| \leq M < L \tag{4.25}$$

if (4.23) holds. Similarly, if

$$u^+(x_0) < u(x_0), \tag{4.26}$$

then

$$|Du(x_0)| \leq M < L$$

for

$$M := L - \frac{u(x_0) - u^+(x_0)}{d_U(x_0, \partial U)}.$$

For every point $x_0 \in U - T$ either (4.23) or (4.26) holds: assertion (4.22) is proved.

2. Now take $v = u$ in (4.17):

$$\int_U |Du^k - Du|^2 \, d\sigma^k \to 0.$$

Fix $\eta > 0$ and write $A_\eta := \{x \in U \mid Du(x) \text{ exists}, |Du(x)| \leq L - \eta\}$. Then

$$\int_{A_\eta} |Du^k - Du|^2 \, d\sigma^k \to 0.$$

Hence

$$\begin{aligned}\limsup_{k\to\infty} \int_{A_\eta} |Du^k| \, d\sigma^k &\leq \limsup_{k\to\infty} \int_{A_\eta} |Du| \, d\sigma^k + \int_{A_\eta} |Du^k - Du| \, d\sigma^k \\ &\leq (L - \eta) \limsup_{k\to\infty} \sigma^k(A_\eta).\end{aligned}$$

Since $\sigma^k\{|Du^k| \leq L - \varepsilon\} \to 0$ for each $\varepsilon > 0$, we deduce

$$L \limsup_{k\to\infty} \sigma^k(A_\eta) \leq (L - \eta) \limsup_{k\to\infty} \sigma^k(A_\eta).$$

Consequently,

$$\lim_{k\to\infty} \sigma^k(A_\eta) = 0 \qquad \text{for each } \eta > 0.$$

Next, define

$$B_\eta := \{x \in U \mid S(x) < L - \eta\} \cap (U - T),$$

the function S defined by (1.5). Since S is upper semicontinuous, B_η is an open set; and hence

$$\sigma(B_\eta) \leq \liminf_{k\to\infty} \sigma^k(B_\eta) \leq \liminf_{k\to\infty} \sigma^k(A_\eta) = 0.$$

Therefore

$$\sigma(U - T) = \sum_{k=1}^{\infty} \sigma(B_{\frac{1}{k}}) = 0.$$

2. Recall that $u^+ \leq u \leq u^-$, with equality along T. Now u^- is locally semiconcave within U, meaning that for each subregion $V \subset\subset U$, there exists a constant C such that

$$D^2 u^- \leq CI \qquad \text{within } V$$

in the sense of distributions. Likewise u^+ is locally semiconvex. It follows that if $x_0 \in T \cap U$, then

$$u(x) = u(x_0) + Du(x_0) \cdot (x - x_0) + O(|x - x_0|^2)$$

for

$$Du(x_0) := Du^-(x_0) = Du^+(x_0).$$

$\square$

Theorem 4.4. *For each $\phi \in C^1_c(U)$, we have the integral identity*

$$\int_U D\phi \cdot Du \, d\sigma = 0. \tag{4.27}$$

We interpret (4.27) as saying

$$\text{“}\operatorname{div}(\sigma Du) = 0 \quad \text{within } U\text{”}$$

in the sense of distributions. Since $Du(x)$ exists for each $x \in \operatorname{spt}(\sigma) \cap U$, the integrand in (4.27) is defined.

Proof. 1. Let $V \subset\subset U$. Take a smooth cutoff function ζ such that

$$0 \leq \zeta \leq 1, \ \zeta \equiv 1 \ \text{ on } V, \ \zeta \equiv 0 \ \text{ near } \partial U.$$

Extend $u = g$ outside U and write

$$u^\varepsilon = \eta_\varepsilon * u,$$

where η_ε is a standard mollifier. Then (4.7) implies

$$0 = \int_U Du^k \cdot D(\zeta(u^k - u^\varepsilon)) \, d\sigma^k;$$

and so

$$\int_U \zeta Du^k \cdot (Du^k - Du^\varepsilon) d\sigma^k = -\int_U Du^k \cdot D\zeta (u^k - u^\varepsilon) \, d\sigma^k.$$

Rewriting the term on the left as in the proof of Lemma 4.2, we find

$$\int_U \zeta |Du^k - Du^\varepsilon|^2 \, d\sigma^k + \int_U \zeta Du^k \cdot Du^\varepsilon \, d\sigma^k = \int_U \zeta |Du^\varepsilon|^2 \, d\sigma^k - \int_U Du^k \cdot D\zeta (u^k - u^\varepsilon) d\sigma^k.$$

Hence

$$\begin{aligned}
&\int_U \zeta |Du^k - Du^\varepsilon|^2 \, d\sigma^k + \int_U \zeta Du^k \cdot Du \, d\sigma^k \\
&\quad = \int_U \zeta (Du - Du^\varepsilon) \cdot Du^k \, d\sigma^k + \int_U \zeta |Du^\varepsilon|^2 \, d\sigma^k - \int_U Du^k \cdot D\zeta (u^k - u^\varepsilon) d\sigma^k \\
&\quad = -\int_U (u - u^\varepsilon) D\zeta \cdot Du^k \, d\sigma^k + \int_U \zeta |Du^\varepsilon|^2 \, d\sigma^k - \int_U Du^k \cdot D\zeta (u^k - u^\varepsilon) d\sigma^k;
\end{aligned}$$

and then

$$\begin{aligned}
&\int_U \zeta |Du^k - Du^\varepsilon|^2 \, d\sigma^k + \int_U \zeta |Du^k|^2 \, d\sigma^k \\
&\quad = \int_U \zeta (Du^k - Du) \cdot Du^k \, d\sigma^k - \int_U (u - u^\varepsilon) D\zeta \cdot Du^k \, d\sigma^k \\
&\qquad + \int_U \zeta |Du^\varepsilon|^2 \, d\sigma^k - \int_U Du^k \cdot D\zeta (u^k - u^\varepsilon) \, d\sigma^k.
\end{aligned}$$

Now $|Du^\varepsilon| \le L$ and $\int_U |Du^k|^2\, d\sigma^k \le C$. Hence

$$\limsup_{k\to\infty} \int_U \zeta |Du^k - Du^\varepsilon|^2 d\sigma^k + L^2 \int_U \zeta\, d\sigma \le L^2 \int_U \zeta\, d\sigma + C \sup_U |u - u^\varepsilon|.$$

Consequently,

$$\limsup_{k\to\infty} \int_V |Du^k - Du^\varepsilon|^2 d\sigma^k \le C\varepsilon. \tag{4.28}$$

2. Let $\phi \in C_c^1(U)$ and take $\operatorname{spt}(\phi) \subset V \subset\subset U$ for some open set V, as above. Then (4.7) implies

$$\begin{aligned} 0 &= \int_U D\phi \cdot Du^k\, d\sigma^k \\ &= \int_U D\phi \cdot (Du^k - Du^\varepsilon)\, d\sigma^k + \int_U D\phi \cdot Du^\varepsilon\, d\sigma^k. \end{aligned}$$

According then to (4.28) we have

$$\left| \int_U D\phi \cdot Du^\varepsilon d\sigma \right| \le C\varepsilon^{1/2}.$$

But $Du^\varepsilon \to Du$ pointwise on $\operatorname{spt}(\sigma)$, and thus the Dominated Convergence Theorem implies

$$\int_U D\phi \cdot Du\, d\sigma = 0.$$

□

4.3 Boundary behavior. Since our test function ϕ is required to have compact support within U, Theorem 4.4 says nothing interesting when

$$\sigma(U) = 0,\ \sigma(\partial U) = 1, \tag{4.29}$$

that is, when σ concentrates all its mass into the boundary of U. We propose in this case the interpretation that

$$\text{“}\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \operatorname{spt}(\sigma)\text{”} \tag{4.30}$$

To see this, write $\phi(x) := \operatorname{dist}(x, \partial U)$ and introduce a smooth vecor field $\boldsymbol{\nu} = (\nu^1, \dots, \nu^n)$ such that $\boldsymbol{\nu} = -D\phi$ near ∂U. In particular, $\boldsymbol{\nu}|_{\partial U}$ is the outward pointing unit normal.

Lemma 4.5. *Assume* (4.29). *Then*

$$\lim_{k\to\infty} \int_U \left(\frac{\partial u^k}{\partial \nu}\right)^2 d\sigma^k = 0, \tag{4.31}$$

where

$$\frac{\partial u^k}{\partial \nu} := Du^k \cdot \boldsymbol{\nu}.$$

Proof. 1. We can rewrite the Euler–Lagrange equation (4.7) to read:

$$(\sigma^k(\delta_{ij} - k u^k_{x_i} u^k_{x_j}))_{x_i} = 0 \tag{4.32}$$

for $j = 1, \dots, n$. For $\varepsilon > 0$ define

$$U_\varepsilon := \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}.$$

We multiply (4.32) by $\nu^j \phi$, sum on j and integrate by parts:

$$\int_U \left(\frac{1}{k}\delta_{ij} - u^k_{x_i} u^k_{x_j}\right)(\nu^j_{x_i}\phi + \nu^j \phi_{x_i})\, d\sigma^k = 0.$$

Now $\phi_{x_i} = -\nu^i$ near ∂U. Hence for $\varepsilon > 0$ small enough,

$$\begin{aligned}\int_U \left(\frac{\partial u^k}{\partial \nu}\right)^2 d\sigma^k &\le \frac{C}{k} + C\int_{U_\varepsilon} |Du^k|^2\, d\sigma^k + C\int_{U - U_\varepsilon} |Du^k|^2 \phi\, d\sigma^k \\ &\le \frac{C}{k} + C\int_{U_\varepsilon} |Du^k|^2\, d\sigma^k + \varepsilon C \int_U |Du^k|^2 d\sigma^k.\end{aligned} \tag{4.33}$$

2. We claim next that for each fixed $\varepsilon > 0$,

$$\lim_{k\to\infty} \int_{U_\varepsilon} |Du^k|^2\, d\sigma^k = 0. \tag{4.34}$$

To confirm this, multiply (4.7) by $\zeta^2 u^k$ and integrate by parts, where $\zeta \in C^\infty_c(U)$. We find

$$\begin{aligned}\int_U |Du^k|^2 \zeta^2\, d\sigma^k &= -2\int_U Du^k \cdot D\zeta\, \zeta u^k\, d\sigma^k \\ &\le \frac{1}{2}\int_U |Du^k|^2 \zeta^2\, d\sigma^k + C\int_U (u^k)^2 |D\zeta|^2 d\sigma^k.\end{aligned}$$

Take ζ so that $\zeta \equiv 1$ on U_ε, $\zeta \equiv 0$ on $U - U_{\varepsilon/2}$. Then

$$\int_{U_\varepsilon} |Du^k|^2\, d\sigma^k \le C\int_{U_{\varepsilon/2}} |D\zeta|^2 d\sigma^k,$$

and the term on the right goes to zero as $k \to \infty$, owing to (4.29). This proves (4.34).

2. Select $\delta > 0$ and then fix small $\varepsilon > 0$ so that $L^2\varepsilon \le \delta$. Using (4.33) and (4.34), we calculate

$$\begin{aligned}\limsup_{k\to\infty} \int_U \left(\frac{\partial u^k}{\partial \nu}\right)^2 d\sigma^k &\le \limsup_{k\to\infty} \int_{U - U_\varepsilon} \left(\frac{\partial u^k}{\partial \nu}\right)^2 d\sigma^k + \lim_{k\to\infty} \int_{U_\varepsilon} |Du^k|^2\, d\sigma^k \\ &\le \varepsilon L^2 = \delta.\end{aligned}$$

□

5. Compression and expansion of characteristics.

We return now to the approximating problem (4.2), or equivalently (4.3):

$$-u^k_{x_i} u^k_{x_j} u^k_{x_i x_j} - \frac{1}{k}\Delta u^k = 0 \quad \text{in } U, \tag{5.1}$$

and discuss the corresponding "characteristics". By these we mean trajectories of the solution $\mathbf{\Phi}^k = \mathbf{\Phi}^k(x,t)$ of the ODE flow

$$\begin{cases} \mathbf{\Phi}^k_t = Du^k(\mathbf{\Phi}^k) \\ \mathbf{\Phi}^k(x,0) = x, \end{cases} \tag{5.2}$$

defined for $x \in U$ and those times t such that $\mathbf{\Phi}^k(x,t) \in U$.

We write

$$J^k(x,t) := \det D_x \mathbf{\Phi}^k(x,t) > 0 \tag{5.3}$$

for the Jacobian of the transformation induced by (5.2), and recall that

$$J^k_t = \operatorname{div}(Du^k) J^k = \Delta u^k J^k. \tag{5.4}$$

We next calculate how u^k and $|Du^k|^2$ change along the flow lines:

Theorem 5.1. *We have these indentities:*

$$\frac{d}{dt} u^k(\mathbf{\Phi}^k) = |Du^k|^2(\mathbf{\Phi}^k) \tag{5.5}$$

and

$$\frac{d^2}{dt^2} u^k(\mathbf{\Phi}^k) = \frac{d}{dt}|Du^k|^2(\mathbf{\Phi}^k) = -\frac{2}{k}\frac{d}{dt}(\log J^k). \tag{5.6}$$

In particular,

$$\frac{1}{2}|Du^k|^2(\mathbf{\Phi}^k(x,t)) + \frac{1}{k}\log J^k(x,t) \equiv \frac{1}{2}|Du^k|^2(x), \tag{5.7}$$

for $x \in U$ and those t such that $\mathbf{\Phi}^k(x,t) \in U$.

Proof. Formula (5.5) is obvious; and (5.6) follows from (5.1) and (5.4), since

$$\frac{d}{dt}|Du^k|^2(\mathbf{\Phi}^k) = 2u^k_{x_i} u^k_{x_i,x_j} u^k_{x_j} = 2\Delta_\infty u^k = -\frac{2}{k}\Delta u^k = -\frac{2}{k}\frac{d}{dt}(\log J^k).$$

$\square$

Interpretation: compression and expansion of characteristics. According to (5.7) the Jacobian of the transformation induced by the ODE (5.2) is given by

$$J^k(x,t) = e^{\frac{k}{2}(|Du^k|^2(x) - |Du^k|^2(\mathbf{\Phi}^k(x,t))}. \tag{5.8}$$

In particular, if $t \mapsto |Du^k|^2(\mathbf{\Phi}^k(x,t))$ is strictly increasing (or decreasing), then J^k goes to zero (or infinity) exponentially fast as $k \to \infty$. And this in turn means geometrically that the flow lines are compressing (or expanding) exponentially fast.

An example. This observation helps "explain" some puzzling phenomena for a simple solution in two dimensions found by Aronsson:

$$u(x_1, x_2) = x_2^{\frac{4}{3}} - x_1^{\frac{4}{3}}.$$

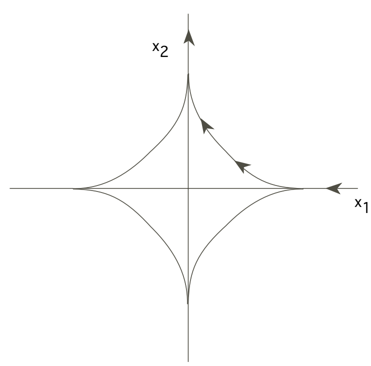
Now the "characteristic" ODE

$$\begin{cases} \mathbf{\Phi}_t = Du(\mathbf{\Phi}) \\ \mathbf{\Phi}(x,0) = x, \end{cases}$$

for $x = (x_1, x_2)$ correspond to trajectories lying on the curves

$$x_1^{\frac{2}{3}} + x_2^{\frac{2}{3}} \equiv C$$

for positive constants C. In the four open quadrants, we see that $|Du|^2$ is indeed constant along these curves, which however run into the x_1 and x_2 axes, as illustrated.



Trajectory of "Characteristics"

We can nonrigorously interpret what is happening here. Consider intial points $x = (x_1, x_2)$ first quadrant. Presumably for large values of k the trajectories of the ODE (5.2) approach the positive x_2 axis, where they "bunch up", corresponding to the Jacobian J^k becoming very small. In view of (5.8) this means that $|Du^k|^2$ then becomes larger than its initial value along the trajectory.

6. More about solutions.

This final section records various observations about the behavior of particular solutions of the infinity-Laplacian equation in certain special circumstances.

6.1 Symmetry and gradient decay. We begin with an assertion about the geometric decay of the sup-norm of $|Du|$, under the hypothesis that the solution u is even:

Theorem 6.1. (i) *Let $u \in C^{0,1}(B(0,1))$ be a viscosity solution of*

$$-\Delta_\infty u = 0 \quad \text{in } B^0(0,1) \tag{6.1}$$

satisfying the symmetry condition:

$$u(x) = u(-x) \qquad \text{for } x \in B(0,1). \tag{6.2}$$

Then there exists a constant $0 < \lambda < 1$, independent of u, such that

$$\text{ess-sup}_{B(0,\frac{1}{2})} |Du| \le \lambda \, \text{ess-sup}_{B(0,1)} |Du|. \tag{6.3}$$

(ii) *If u satisfies the above conditions and*

$$\text{ess-sup}_{B(0,1)} |Du| \le 1,$$

then

$$|Du(x)| \le \frac{1}{\lambda}|x|^\alpha \ \text{ and } \ |u(x)| \le u(0) + \frac{1}{\lambda(1+\alpha)}|x|^{1+\alpha},$$

for $x \in B(0,1)$, where $\alpha := \log_2 \frac{1}{\lambda}$.

Proof. 1. The proof is by contradiction. Suppose that for each j there exists a viscosity solution $u_j \in C^{0,1}(B(0,1))$ of (6.1), satisfying (6.2), such that

$$\text{ess-sup}_{B(0,1)} |Du_j| = 1, \tag{6.4}$$

but

$$\text{ess-sup}_{B(0,\frac{1}{2})} |Du_j| \ge 1 - \frac{1}{j}. \tag{6.5}$$

We may assume $u_j(0) = 0$ for all j, and also

$$\lim_{j\to\infty} u_j = u_\infty \quad \text{uniformly on } B(0,1).$$

According to (6.4), $\text{ess-sup}_{B(0,1)} |Du_\infty| \le 1$.

2. We claim that in fact

$$\text{ess-sup}_{B(0,\frac{1}{2})}\,|Du_\infty| = 1.$$

To see this, note first that owing to (6.5) for each j there exists a point $x_j \in B(0,\frac{1}{2})$ such that

$$S^+(u_j, x_j) \geq 1 - \frac{2}{j}.$$

Hence for each $0 < r < \frac{1}{2}$, we have

$$S^+(u_j, x_j, r) \geq 1 - \frac{2}{j}.$$

Let us assume

$$\lim_{j\to\infty} x_j = x_\infty \in B(0,\tfrac{1}{2}).$$

Then

$$\lim_{j\to\infty} S^+(u_j, x_j, r) = S^+(u_\infty, x_\infty, r).$$

Hence $S({u_\infty}^+, x_\infty, r) \geq 1$ for all radii $0 < r < \frac{1}{2}$, and consequently

$$S^+(u_\infty, x_\infty) \geq 1.$$

But since $\text{ess-sup}_{B(0,1)}\,|Du_\infty| \leq 1$, we must have

$$1 = S({u_\infty}^+, x_\infty) = \text{ess-sup}_{B(0,1)}\,|Du_\infty|.$$

Therefore u is differentiable at x_∞ and

$$|Du_\infty(x_\infty)| = S^+(u_\infty, x_\infty) = 1.$$

3. Moreover, there exists a parameterized line segment $\mathbf{l} : [-b, a] \to B(0,1)$ such that

$$\mathbf{l}'(t) = Du_\infty(x_\infty),\ \mathbf{l}(0) = x_\infty$$

and

$$\mathbf{l}(-b),\ \mathbf{l}(a) \in \partial B(0,1).$$

Let $x^+ = \mathbf{l}(a)$ and $x^- = \mathbf{l}(-b)$. Then

$$\frac{|u(x^+) - u(x^-)|}{|x^+ - x^-|} = 1.$$

It is easy to see that

$$|x^+ + x^-| \leq 1.$$

Therefore

$$|u(x^+) - u(x^-)| = |u(x^+) - u(-x^-)| \leq |x^+ + x^-| \leq 1.$$

Moreover,

$$|x^+ - x^-| \geq 2\sqrt{1 - \frac{1}{4}} = \sqrt{3};$$

and consequently

$$\frac{|u(x^+) - u(x^-)|}{|x^+ - x^-|} \leq \frac{1}{\sqrt{3}} < 1,$$

a contradiction. □

6.2 Effects of boundary conditions. We have found it difficult to develop much intuition about the behavior of infinity harmonic functions, and have consequently found it interesting to study some relatively simple situtations.

Here is a good question: Suppose that u is nonnegative, bounded and solves $-\Delta_\infty u = 0$ in the unit ball $B(0,1)$, with boundary values that vanish everywhere except near the north pole of $\partial B(0,1)$, where $u = 1$. We now shrink the region on the boundary near the north pole where $u \neq 0$, and ask: Does $u(0)$ go to zero or not?

Certainly for harmonic functions, $u(0)$ would go to zero. But this assertion is not so obvious for infinity harmonic functions. A. Oberman [O] has provided some careful computations showing that $u(0)$ goes indeed go to zero, a fact we next prove:

Theorem 6.2. *For each m, let $u_m \in C(B(0,1))$ be a viscosity solution of*

$$-\Delta_\infty u_m = 0 \quad \text{in } B^0(0,1)$$

satisfying

$$0 \leq u_m \leq 1, \quad u_m|_{A_m} = 0,$$

where

$$A_m := \{x \in \partial B(0,1) \mid \ -1 \leq x_n \leq 1 - \frac{1}{m}\}.$$

Then

$$\lim_{j \to \infty} u_m = 0 \quad \text{uniformly on compact subsets of } B(0,1) - \{e_n\},$$

for $e_n = (0, ..., 0, 1)$.

Proof. For each m, set

$$R_m := \{x \in \mathbb{R}^n \mid \ -1 < x_i < 1 \text{ for } 1 \leq i \leq n-1, \ -1 < x_n < 1 - \frac{1}{m}\}$$

and

$$I_m := \partial R_m \cap B(0,1).$$

Choose v_m solving

$$\begin{cases} -\Delta_\infty v_m = 0 \quad \text{in } R_m \\ v_m|_{\partial R_m - I_m} = 0, \quad v_m|_{I_m} = u_m. \end{cases}$$

According to [C-E-G],

$$|Dv_m(x)| \leq \frac{1}{\operatorname{dist}(x, I_m)} \quad \text{for a.e } x \in R_m.$$

Hence, without loss of generality, we may assume

$$\lim_{m\to\infty} v_m = v \quad \text{uniformly on compact subsets of } \bar{R} - \{e_n\},$$

where

$$R = \{x \in \mathbb{R}^n | -1 < x_i < 1 \ \text{ for } \ 1 \leq i \leq n\}$$

and $v \in C(\bar{R} - \{e_n\})$ is a viscosity solution of

$$-\Delta_\infty v = 0 \quad \text{in } R$$

satisfying

$$0 \leq v \leq 1, \quad v|_{\partial R - \{e_n\}} = 0.$$

We can now quote a nice removable singularity result of Bhattacharya [Bh], namely that

$$v \equiv 0 \quad \text{in } \bar{R}.$$

Since

$$v_m|_{R_m \cap \partial B(0,1)} \geq 0 = u_m|_{R_m \cap \partial B(0,1)}, \quad v_m|_{I_m} = u_m,$$

it follows by comparison that

$$v_m|_{\partial R_m} \geq u_m.$$

Hence

$$v_m|_{\bar{R}_m \cap B(0,1)} \geq u_m|_{\bar{R}_m \cap B(0,1)}.$$

□

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