Envelopes and nonconvex Hamilton–Jacobi equations

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Abstract

This paper introduces a new representation formula for viscosity solutions of nonconvex Hamilton–Jacobi PDE using "generalized envelopes" of affine solutions. We study as well envelope and singular characteristic constructions of equivocal surfaces and discuss also differential game theoretic interpretations.

In memory of Arik A. Melikyan.

1 Introduction

This paper is devoted to several themes, mostly concerning envelope constructions of viscosity solutions for nonconvex Hamilton–Jacobi PDE and of their singular surfaces. We explain as well some implications for Hamilton–Jacobi–Isaacs PDE from differential game theory.

1.1 Constructing solutions from envelopes. Given a smooth Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$ and a smooth function $g : \mathbb{R}^n \to \mathbb{R}$, define

$$v(x, y, t) := g(y) + (x - y) \cdot Dg(y) - tH(Dg(y)). \tag{1.1}$$

Then for each fixed $y \in \mathbb{R}^n$, the mapping $(x, t) \mapsto v(x, y, t)$ is affine and solves the Hamilton–Jacobi equation $u_t + H(Du) = 0$. We compute the envelope of this family of functions by setting

$$D_y v(x, y, t) = (x - y - tDH(Dg(y)))D^2 g(y) = 0. \tag{1.2}$$

Assuming $D^2 g$ is nonsingular, we see that therefore

$$x = y + tDH(Dg(y)). \tag{1.3}$$

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For the moment, let us suppose we can uniquely and smoothly solve this expression for $y = y(x,t)$. The envelope of the family of functions v is then $u(x,t) := v(x, y(x,t), t)$; and (1.3) implies

$$u_t(x,t) = -H(Dg(y(x,t))), \quad Du(x,t) = Dg(y(x,t)). \tag{1.4}$$

Consequently, the envelope u so defined solves the initial-value problem

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{1.5}$$

These computations are of course purely formal, but motivate the following rigorous considerations.

Envelopes and viscosity solutions. Lax, E. Hopf and other authors long ago understood for a convex Hamiltonian H that the expression

$$u(x,t) := \inf_{y\in\mathbb{R}^n} \left\{ g(y) + tL\left(\tfrac{x-y}{t}\right)\right\} \qquad \text{(Hopf–Lax)} \tag{1.6}$$

where $L := H^*$ is the dual convex function of H, provides the correct solution of (1.5), in modern parlance the viscosity solution. This is in fact a two-parameter envelope construction, as we can rewrite (1.6) as

$$u(x,t) := \inf_{y\in\mathbb{R}^n} \sup_{z\in\mathbb{R}^n} \{g(y) + (x-y)\cdot z - tH(z)\}. \tag{1.7}$$

E. Hopf in [H] proposed a second formula for nonconvex Hamiltonians, but with a convex initial function g that grows at most linearly:

$$u(x,t) := \sup_{z\in\mathbb{R}^n} \{x\cdot z - g^*(z) - tH(z)\} \qquad \text{(Hopf)} \tag{1.8}$$

where g^* denotes the convex dual of g. This is also a two-parameter envelope, had by interchanging the sup and inf in (1.7):

$$u(x,t) := \sup_{z\in\mathbb{R}^n} \inf_{y\in\mathbb{R}^n} \{g(y) + (x-y)\cdot z - tH(z)\}. \tag{1.9}$$

Observe that our extremizing the two-parameter family of solutions $g(y)+(x-y)\cdot z-tH(z)$ in y implies that $z = Dg(y)$, in which case the one-parameter family of solutions (1.1) appears.

Inspired by Osher's paper [O], M. Bardi and I proved in [B-E] that (1.8) provides a representation formula for the unique viscosity solution of the initial value problem (1.5), provided that g is convex. See Lions–Rochet [L-R] or Cardaliaguet [C] for better proofs; and look also at Theorem 3.2 below. An interesting related paper is Bardi–Faggian [B-F].

Section 2 of this paper derives a new representation formula for the viscosity solution of (1.5), for nonconvex H and nonconvex initial data g, as a sort-of generalized envelope of the affine solutions (1.1). We discuss various applications in Section 3, and in particular interpret our expression as a generalization of both the Hopf–Lax and Hopf formulas. The main technical tool is introducing solutions σ^ϵ to the adjoint of the formal linearization of a smooth approximation to (1.5), a method I used for nonconvex Hamilton–Jacobi PDE in a different way in my earlier paper [E]. See also Tran [T] and Cagnetti–Gomes–Tran [C-G-T].

1.2 Equivocal surfaces as envelopes. A big difference between convex versus nonconvex Hamiltonians H is that for the latter the viscosity solution u of the initial-value problem (1.5) can admit what we will call in this paper "envelope shocks", that is, surfaces of discontinuity of ∇u from which characteristics can leave tangentially going forward in time: see Figure 1.

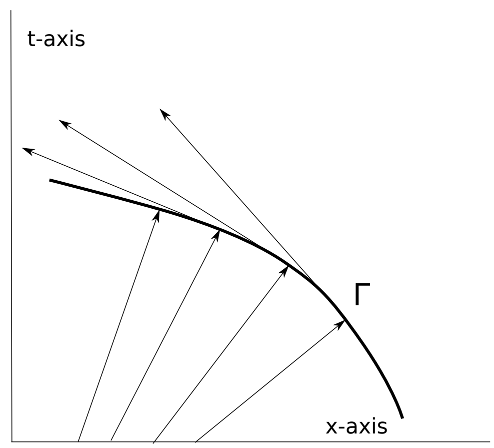

Figure 1: An "envelope shock"

An important research goal is understanding how the nonconvexity of H gives rise to such structures. As a step towards this, we explain in Section 4 some explicit expressions for these envelope shocks in terms of the gradient of u on both sides and the second derivatives of u from the "incoming" side. These calculations provide simplified derivations of general formulas found by A. Melikyan in his extremely interesting book [M] on singular characteristics.

1.3 Differential games. Two-person zero-sum differential games provide a rich variety of Hamilton-Jacobi type PDE with nonconvex Hamiltonians. In particular, Isaacs [I, Chapter 10] identified for certain differential games what he called "equivocal surfaces", which are singular surfaces for the corresponding HJI equation, along which one or both players has many optimal strategies available. Differential games may also entail barriers, which are surfaces of discontinuity of the value function u. In Sections 5 and 6 we present a PDE interpretation of barriers and a game theoretic interpretation of envelope shocks as equivocal surfaces. We end with some new geometric insights into an example due to Isaacs.

We hereafter informally use the fortuitous abbreviation ES for both "equivocal surfaces" and "envelope shocks".

Notation. We use "D" for the gradient of a function of the n variables $x = (x_1, \dots, x_n)$, and "∇" for the full gradient of a function of the $n+1$ variables $(x,t) = (x_1, \dots, x_n, t)$. Thus $\nabla u = (Du, u_t)$. We will also often denote a point of $\mathbb{R}^{n+1}$ as $q = (p, p_{n+1})$ for $p \in \mathbb{R}^n$ and $p_{n+1} \in \mathbb{R}$.

2 A representation formula

In this section we set forth for nonconvex H an envelope-type representation formula for the unique viscosity solution of the initial value problem (1.5). We will later see that this expression generalizes Hopf's formula (1.8) to nonconvex initial data (but is much less explicit).

2.1 An integral formula in terms of plane waves. Assume for now that $H : \mathbb{R}^n \to \mathbb{R}$ is C^1 and $g : \mathbb{R}^n \to \mathbb{R}$ is bounded and C^1, with bounded gradient. Let u denote the unique viscosity solution of the initial value problem (1.5).

THEOREM 2.1 *For $\mathcal{L}^{n+1}$ almost every point $(x,t) \in \mathbb{R}^n \times (0,\infty)$ there exists a Radon probability measure $\gamma_{x,t}$ on $\mathbb{R}^n$ such that*

$$u(x,t) = \int_{\mathbb{R}^n} g(y) + (x-y) \cdot Dg(y) - tH(Dg(y))\, d\gamma_{x,t}. \tag{2.1}$$

Furthermore,

$$u_t(x,t) = -\int_{\mathbb{R}^n} H(Dg(y))\, d\gamma_{x,t}, \quad Du(x,t) = \int_{\mathbb{R}^n} Dg(y)\, d\gamma_{x,t}, \tag{2.2}$$

and

$$H\left(\int_{\mathbb{R}^n} Dg(y)\, d\gamma_{x,t}\right) = \int_{\mathbb{R}^n} H(Dg(y))\, d\gamma_{x,t}. \tag{2.3}$$

Observe that although the expression (2.1) may appear to be linear, it is in fact highly nonlinear, since the measure $\gamma_{x,t}$ depends upon the solution. Our representation formula is rather a sort-of "linear/envelope decomposition" of the solution u into the plane wave solutions (1.1). Note in particular that the expressions (2.2) generalize the formulas (1.4), and indeed coincide with (1.4) if $\gamma_{x,t} = \delta_{y(x,t)}$. In general $\gamma_{x,t}$ need not be a point mass, although the nontrivial identity (2.3) must still hold.

2.2 Approximations, estimates. To establish (2.1), (2.2) we consider first the corresponding initial-value problem for the regularized Hamilton–Jacobi equation:

$$\begin{cases} u_t^\varepsilon + H(Du^\varepsilon) = \varepsilon \Delta u^\varepsilon & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\varepsilon = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{2.4}$$

As in [E] (see also Tran [T]) we study the adjoint of the linearization of (2.4). Given next a time $t_1 > 0$ and a Radon probability measure α on $\mathbb{R}^n$, we introduce this terminal value problem, the adjoint of the linearization of (2.4):

$$\begin{cases} -\sigma_t^\varepsilon - \operatorname{div}(\sigma^\varepsilon DH(Du^\varepsilon)) = \varepsilon \Delta \sigma^\varepsilon & \text{in } \mathbb{R}^n \times [0,t_1) \\ \sigma^\varepsilon = \alpha & \text{on } \mathbb{R}^n \times \{t=t_1\}. \end{cases} \tag{2.5}$$

We record next the elementary estimates that for each time $t_1 > 0$

$$\sup_{\mathbb{R}^n \times [0,t_1]} |u^\varepsilon|, |Du^\varepsilon|, |u_t^\varepsilon| \le C \quad \text{and} \quad \sigma^\varepsilon \ge 0, \int_{\mathbb{R}^n} \sigma^\varepsilon \, dx = 1 \quad (0 \le t < t_1). \tag{2.6}$$

Also, as demonstrated in [E] there exists a constant C, independent of $\varepsilon > 0$, such that

$$\varepsilon \int_0^{t_1} \int_{\mathbb{R}^n} (|D^2 u^\varepsilon|^2 + |Du_t^\varepsilon|^2) \sigma^\varepsilon dx dt \le C. \tag{2.7}$$

Furthermore, $u^\varepsilon \to u$ locally uniformly, where u is the unique viscosity solution of (1.5).

2.3 Approximating a point mass by the average over a cylinder. Fix a point $(x_1, t_1) \in \mathbb{R}^n \times (0,\infty)$, and define the cylinder

$$C(x_1, t_1, r) := B(x_1, r) \times [t_1, t_1 + r] \tag{2.8}$$

for small $r > 0$. Fix $r > 0$ and let $\sigma_{s,r}^\varepsilon$ denote the solution of (2.5) having as terminal data the function

$$\alpha_r := \frac{1}{|B(x_1,r)|} \chi_{B(x_1,r)}. \tag{2.9}$$

at time s, where $t_1 \le s \le t_1 + r$. Now average with respect to s:

$$\sigma_r^\varepsilon(x,t) := ⨍_{t_1}^{t_1+r} \sigma_{s,r}^\varepsilon(x,t) \, ds \tag{2.10}$$

for $(x,t) \in \mathbb{R}^n \times [0, t_1)$; and observe that σ_r^ε solves the adjoint PDE (2.5).

THEOREM 2.2 (i) *The function*

$$w^\varepsilon := u^\varepsilon - x \cdot Du^\varepsilon - tu^\varepsilon_t \tag{2.11}$$

solves the PDE

$$w^\varepsilon_t + DH(Du^\varepsilon) \cdot Dw^\varepsilon = \varepsilon \Delta w^\varepsilon + \varepsilon \Delta u^\varepsilon. \tag{2.12}$$

(ii) *Also,*

$$\fint_{C(x_1,t_1,r)} w^\varepsilon \, dxdt = \int_{\mathbb{R}^n} w^\varepsilon(y,0)\sigma^\varepsilon_r(y,0)\, dy + O(\varepsilon^{\frac{1}{2}}). \tag{2.13}$$

Proof. 1. We have $w^\varepsilon_t = -x \cdot Du^\varepsilon_t - tu^\varepsilon_{tt}$, $w^\varepsilon_{x_i} = -x \cdot Du^\varepsilon_{x_i} - tu^\varepsilon_{x_i t}$, and therefore

$$\begin{aligned} w^\varepsilon_t + H_{p_i} w^\varepsilon_{x_i} &= -x \cdot (Du^\varepsilon_t + H_{p_i} Du^\varepsilon_{x_i}) - t(u^\varepsilon_{tt} + H_{p_i} Du^\varepsilon_t) \\ &= \varepsilon(-x \cdot \Delta Du^\varepsilon - t\Delta u^\varepsilon_t) = \varepsilon \Delta w^\varepsilon + \varepsilon \Delta u^\varepsilon. \end{aligned}$$

2. Select a time $t_1 \le s \le t_1 + r$, and define $\sigma^\varepsilon_{s,r}$ as above. Then for times $0 \le t \le s$, we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} w^\varepsilon \sigma^\varepsilon_{s,r}\, dx &= \int_{\mathbb{R}^n} w^\varepsilon_t \sigma^\varepsilon_{s,r} + w^\varepsilon (\sigma^\varepsilon_{s,r})_t \, dx \\ &= \int_{\mathbb{R}^n} (-H_{p_i} w^\varepsilon_{x_i} + \varepsilon \Delta w^\varepsilon + \varepsilon \Delta u^\varepsilon)\sigma^\varepsilon_{s,r} + w^\varepsilon(-(\sigma^\varepsilon_{s,r} H_{p_i})_{x_i} - \varepsilon \Delta \sigma^\varepsilon_{s,r})\, dx \\ &= \int_{\mathbb{R}^n} \varepsilon \Delta u^\varepsilon \sigma^\varepsilon_{s,r}\, dx = O(\varepsilon^{\frac{1}{2}}), \end{aligned}$$

according to the estimates (2.6) and (2.7). Integrate from 0 to the time s:

$$\fint_{B(x_1,r)} w^\varepsilon(x,s)\, dx = \int_{\mathbb{R}^n} w^\varepsilon(y,0)\sigma^\varepsilon_{s,r}(y,0)\, dy + O(\varepsilon^{\frac{1}{2}}).$$

Finally, average for $t_1 \le s \le t_1 + r$, to derive the integral identity (2.13). □

2.4 Proof of Theorem 2.1. Since $\int_{\mathbb{R}^n} \sigma^\varepsilon_r(y,0)\, dy = 1$ and $\sigma^\varepsilon_r(y,0) \ge 0$, there exists a sequence $\varepsilon_j \to 0$ such that $\sigma^{\varepsilon_j}_r(y,0)\, dy \rightharpoonup d\sigma_r$ weakly as measures on $\mathbb{R}^n$. Observe next that

$$w^\varepsilon(y,0) = g(y) - y \cdot Dg(y).$$

Since $u^\varepsilon \to u$ locally uniformly and $\nabla u^\varepsilon \rightharpoonup \nabla u$ weakly* in L^∞, we may therefore pass to limits in the identity (2.13), to learn that

$$\fint_{C(x_1,t_1,r)} u - x \cdot Du - tu_t \, dxdt = \int_{\mathbb{R}^n} g(y) - y \cdot Dg(y)\, d\sigma_r.$$

Assume next that (x,t) is a Lebesgue point for ∇u. We now select a sequence $r_j \to 0$ such that $\sigma_{r_j} \rightharpoonup d\gamma_{x,t}$ weakly as measures on $\mathbb{R}^n$. Then our putting $r = r_j$ above and sending $r_j \to 0$ gives

$$u(x,t) - x \cdot Du(x,t) - tu_t(x,t) = \int_{\mathbb{R}^n} g(y) - y \cdot Dg(y)\, d\gamma_{x,t}.$$

We conclude by noting that

$$Du(x,t) = \int_{\mathbb{R}^n} Dg(y)\, d\gamma_{x,t}, \quad u_t(x,t) = \int_{\mathbb{R}^n} u_t(y,0)\, d\gamma_{x,t} = \int_{\mathbb{R}^n} H(Dg(y))\, d\gamma_{x,t}$$

□

Interpretation: an "infinitesimal" domain of dependence. The following manipulations are only formal, but provide some additional understanding.

Given a smooth, bounded function $h : \mathbb{R}^n \to \mathbb{R}$, we suppose for each $-1 < \tau < 1$ that $u^\tau = u^\tau(x,t)$ is the unique viscosity solution of

$$\begin{cases} u^\tau_t + H(Du^\tau) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \qquad\qquad u^\tau = g + \tau h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Then calculations similar to those above suggest this formula for the derivative of u^τ in the parameter τ:

$$\frac{\partial}{\partial \tau} u^\tau(x,t)|_{\tau=0} = \int_{\mathbb{R}^n} h\, d\gamma_{x,t}.$$

In other words, an $O(\tau)$ change in the initial data outside the support of $\gamma_{x,t}$ will have at most a $o(\tau)$ effect on the solution at the point (x,t). □

Remark: unbounded initial data. In one of our applications in Section 3 we will need (2.1) when the initial function g has bounded gradient, but g itself may be unbounded.

Our formula (2.1) is still valid in this case. Indeed, the gradient Du^ϵ is uniformly bounded, and consequently we have finite propagation speed. In particular, as in the proof of Theorem 3.5 (ii) in [E], we can show that $\operatorname{spt} \gamma_{x,t} \subseteq B(x, Mt)$, where $M := \sup\{|DH(Du^\varepsilon)|\}$. So the behavior of the initial data g outside the ball with center x and radius Mt is not relevant.

□

3 Applications

This section discusses some applications of the representation formulas (2.1)–(2.3) and of the adjoint technique used for their proof.

3.1 Homogeneous Hamiltonians. We assume for this subsection that the Hamiltonian H is positively homogeneous of degree one and thus

$$DH(p)\cdot p = H(p) \qquad (p \neq 0). \tag{3.1}$$

Now let u be the unique viscosity solution of the initial-value problem (1.5). Formal calculations using characteristics then imply

$$\dot{\mathbf{x}} = DH(\mathbf{p}),\ \dot{\mathbf{p}} = 0,\ \dot{\mathbf{z}} = \mathbf{p}\cdot DH(\mathbf{p}) - H(\mathbf{p}) = 0,$$

where $\mathbf{p}(t) = Du(\mathbf{x}(t),t)$ and $\mathbf{z}(t) = u(\mathbf{x}(t),t)$; and hence u is constant along the characteristics. The following theorem provides rigorous justification:

THEOREM 3.1 *For $\mathcal{L}^{n+1}$ almost every point $(x,t) \in \mathbb{R}^n \times (0,\infty)$, we have*

$$g = u(x,t) \qquad \gamma_{x,t} \text{ almost everywhere.} \tag{3.2}$$

Proof. In view of (3.1), the solution u^ε of (2.4) satisfies

$$u^\varepsilon_t + DH(Du^\varepsilon)\cdot Du^\varepsilon = \varepsilon\Delta u^\varepsilon \qquad \text{in } \mathbb{R}^n \times (0,\infty).$$

Select a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and put $w^\varepsilon := \phi(u^\varepsilon)$. Then

$$w^\varepsilon_t + DH(Du^\varepsilon)\cdot Dw^\varepsilon = \varepsilon\Delta w^\varepsilon - \varepsilon\phi''(u^\varepsilon)|Du^\varepsilon|^2.$$

As in the previous section, we multiply by σ^ε_r, integrate, and pass to limits, to derive the formula

$$\phi(u(x,t)) = \int_{\mathbb{R}^n} \phi(g(y))\, d\gamma_{x,t}.$$

Now select $\phi(z) := (z - u(x,t))^2$ to finish the proof. □

Our later Theorem 6.1 generalizes this argument to nonhomogeneous Hamiltonians, and provides a game theoretic interpretation.

3.2 The Hopf–Lax and Hopf formulas again. Our representation (2.1), (2.2) and (2.3) generalizes both the classical Hopf–Lax and Hopf formulas (1.6) and (1.8):

THEOREM 3.2 *Suppose u is the unique viscosity solution of* (1.5) *and consequently satisfies* (2.1)–(2.3).

(i) *If H is convex, then the Hopf–Lax expression* (1.6) *is valid for all points $(x,t) \in \mathbb{R}^n \times (0,\infty)$.*

(ii) *If instead g is convex and grows at most linearly, then Hopf's expression* (1.8) *holds for all points $(x,t) \in \mathbb{R}^n \times (0,\infty)$.*

In both cases, for $\mathcal{L}^{n+1}$ almost every point (x,t), $\gamma_{x,t}$ is the unit mass at the point

$$y = x - tDH(Du(x,t)). \tag{3.3}$$

When the solution u has "equivocal surfaces" in the gradient, the measure $\gamma_{x,t}$ is not a point mass. So equivocal surfaces cannot occur for the cases (i) and (ii) of either a convex Hamiltonian or convex initial data.

Proof. 1. For case (i) we may assume H is uniformly convex, since we can approximate by uniformly convex Hamiltonians otherwise.

Select a point (x,t) at which u is differentiable and at which (2.1)–(2.3) hold. Then as H is uniformly convex, it follows from (2.2) and (2.3) that $Dg(y) = Du(x,t)$ for $\gamma_{x,t}$ almost every point y. Since u is differentiable at (x,t), the only possibility is that $\gamma_{x,t}$ is a Dirac mass at the point $y = y(x,t) = x - tH(Du(x,t))$, since otherwise there would be more than one characteristic hitting the point (x,t). Then (2.1) implies

$$\begin{aligned} u(x,t) &= g(y(x,t)) + (x - y(x,t)) \cdot Dg(y(x,t)) - tH(Dg(y(x,t))) \\ &= g(y(x,t)) + tL(\tfrac{x-y(x,t)}{t}), \end{aligned} \tag{3.4}$$

since $DL(\frac{x-y(x,t)}{t}) = Du(x,t) = Dg(y(x,t))$. Also, since $u_t + H(Du) = 0$ implies $u_t + z \cdot Du \leq L(z)$ for each $z \in \mathbb{R}^n$, the comparison principle for viscosity solutions implies

$$u(x,t) \leq g(x - tz) + tL(z).$$

Thus

$$u(x,t) \leq \inf_z \{g(x - tz) + tL(z)\} = \inf_y \{g(y) + tL(\tfrac{x-y}{t})\}.$$

This and (3.4) give (1.6), at those points where (2.1)–(2.3) are valid. By continuity, (1.6) holds for all x, t.

2. Assume now the hypothesis of (ii), and recall from the last remark in Section 2 that our representation formula (2.1) is valid in the current setting.

Let $(x,t) \in \mathbb{R}^n \times (0,\infty)$ be a point at which u is differentiable and at which (2.1) holds. As $\gamma_{x,t}$ is a probability measure, we have

$$\begin{aligned} \sup_z \{x \cdot z - g^*(z) - tH(z)\} &\geq \int_{\mathbb{R}^n} x \cdot Dg(y) - g^*(Dg(y)) - tH(Dg(y))\, d\gamma_{x,t} \\ &= \int_{\mathbb{R}^n} g(y) + (x - y) \cdot Dg(y) - tH(Dg(y))\, d\gamma_{x,t} \\ &= u(x,t), \end{aligned} \tag{3.5}$$

since $g^*(Dg(y)) + g(y) = Dg(y) \cdot y$. Conversely, for each $y, z \in \mathbb{R}^n$ we have $y \cdot z - g^*(z) \leq g(y)$; and consequently the comparison principle for viscosity solutions gives

$$x \cdot z - g^*(z) - tH(z) \leq u(x,t).$$

This inequality is valid for each z. Hopf's representation (1.8) follows, and we see also that the inequality sign in (3.5) is in fact an equality.

Since g grows at most linearly, g^* equals infinity outside some compact set. So the supremum in Hopf's formula (1.8) is really a maximum. Therefore for $\gamma_{x,t}$ almost every y, $z = Dg(y)$ gives the max in Hopf's formula (1.8). But as u is differentiable at the point (x,t), we also have $z = Du(x,t) = Dg(y)$. There is only one possibility for the point $y = x - tH(Du(x,t))$ and thus $\gamma_{x,t}$ is a unit mass there. □

Example: linear-quadratic differential games. The theory of differential games (see Lewin [L, Section 7.4]) provides formulas for very special solutions of quadratic, nonconvex Hamilton–Jacobi PDE. For an easy case, consider the problem

$$\begin{cases} u_t + \frac{1}{2}Du \cdot BDu = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = \frac{1}{2}x \cdot Qx & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{3.6}$$

where B, Q are symmetric $n \times n$ matrices, Q is positive definite, and B has both positive and negative eigenvalues. So the Hamiltonian $H(p) := \frac{1}{2}p \cdot Bp$ is nonconvex, but the initial function $g(x) := \frac{1}{2}x \cdot Qx$ is convex.

Following standard ideas for linear-quadratic problems we seek a solution of the form $u(x,t) = \frac{1}{2}x \cdot S(t)x$, where the symmetric matrix function S verifies the Riccati equation

$$\dot{S} + SBS = 0 \qquad (t > 0) \tag{3.7}$$

with $S(0) = Q$. The solution is $S(t) = (Q^{-1} + tB)^{-1}$, valid until the first time $t^* > 0$ when $Q^{-1} + tB$ fails to be invertible.

Although Hopf's formula (1.8) is not applicable for (3.6), since g grows quadratically, it nevertheless suggests the the alternative representation

$$u(x,t) = \sup_z \{x \cdot z - \tfrac{1}{2}z \cdot Q^{-1}z - \tfrac{t}{2}z \cdot Bz\}. \tag{3.8}$$

For small $t > 0$ the sup occurs for $z = (Q^{-1} + tB)^{-1}x$, and so (3.8) agrees with the answer provided by solving (3.7). The expression (3.8) breaks down at t^* since the supremum becomes infinite. □

3.3 The structure of the envelope shocks. For another application of our representation formulas (2.1) and (2.2), we indicate briefly how the geometry of the characteristic surface

$$\Sigma := \{q \mid p_{n+1} + H(p) = 0\} \tag{3.9}$$

(where $q = (p, p_{n+1})$ for $p \in \mathbb{R}^n$ and $p_{n+1} \in \mathbb{R}$) constrains the complexity of the structure of the characteristics leading to the point (x,t). For this we will assume that u is a piecewise smooth, viscosity solution; and will discuss informally that *we can sometimes estimate, in terms of the location of* $\nabla u(x,t)$ *on* Σ*, the number of times a backwards characteristic starting at* (x,t) *can intersect different envelope surfaces before reaching* $\mathbb{R}^n \times \{t=0\}$.

We assume for this section that u is a Lipschitz continuous and piecewise-C^1 solution. We suppose further that $\lim_{p\to\infty} H(p) = \infty$ and that H is convex outside of some bounded set. Then Σ has the typical shape drawn in Figure 2.

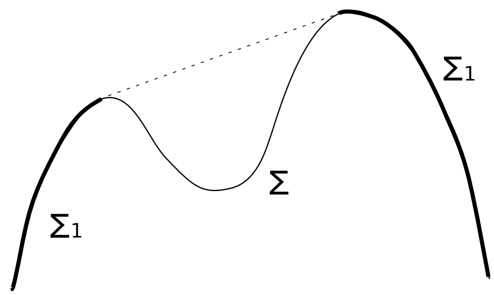

Figure 2: The sets Σ and Σ_1.

The set $\mathbf{\Sigma_1 \subseteq \Sigma}$. To start, let us assume that $\nabla u(x,t)$ lies on

$$\Sigma_1 := \Sigma \cap \operatorname{co}(\Sigma), \tag{3.10}$$

where "co" denote the closed, convex hull.

THEOREM 3.3 *Assume that* u *is differentiable at* (x,t)*. Suppose also that*

$$\nabla u(x,t) \in \Sigma_1 \quad \textit{and } \Sigma_1 \textit{ is strictly convex near } \nabla u(x,t). \tag{3.11}$$

Then $\gamma_{x,t}$ *is the unit mass at the point*

$$y = y(x,t) = x - tDH(Du(x,t)). \tag{3.12}$$

In particular there exists a straight backwards characteristic starting at (x,t) that extends all the way until it hits $\mathbb{R}^n \times \{t=0\}$ at the point y.

Proof. According to (2.2), we have

$$\nabla u(x,t) = \int_{\mathbb{R}^n} (Dg(y), -H(Dg(y)))\, d\gamma_{x,t} = \int_{\mathbb{R}^{n+1}} q\, d\mu_{x,t},$$

where $\mu_{x,t}$ is a probability measure on Σ, the push-forward of $\gamma_{x,t}$ under the mapping $y \mapsto (Dg(y), -H(Dg(y))) \in \Sigma$. Consequently $\nabla u(x,t)$ is the center of mass of a probability measure supported in Σ; and therefore

$$\mu_{x,t} = \delta_{\nabla u(x,t)}, \tag{3.13}$$

since (3.11) and the strict convexity assumption imply that $\nabla u(x,t)$ is an extreme point of the support of $\mu_{x,t}$.

Owing to (3.13), we therefore have $(Dg(y), -H(Dg(y))) = \nabla u(x,t)$ for $\gamma_{x,t}$ almost every point y. Consider now the forward characteristics starting at each such point y in the support of $\gamma_{x,t}$. If and until these characteristics hit the singular set of ∇u, we have $\nabla u = \nabla u(x,t)$ along the path of the characteristic. If one of the characteristics ends at a compressive shock before time t, it cannot therefore reach the point (x,t). Furthermore, the characteristic cannot hit an envelope shock. Thus the only remaining possibility is that there is only one such characteristic, starting at the point $y = x - tH(Du(x,t))$. □

We provide next an alternative interpretation of this result. Let $\hat{H}$ denote the largest convex function less than or equal to H. Then

$$\Sigma_1 = \{q = (p, p_{n+1}) \mid H(p) = \hat{H}(p)\}.$$

We turn our attention to the convex initial-value problem

$$\begin{cases} \hat{u}_t + \hat{H}(D\hat{u}) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \hat{u} = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{3.14}$$

THEOREM 3.4 *Assume that u is differentiable at (x,t) and that (3.11) holds. Then*

$$u(x,t) = \hat{u}(x,t).$$

Proof. Since $\hat{H} \leq H$, we see that $\hat{u}_t + H(D\hat{u}) \geq 0$ in the viscosity sense, and thus $u \leq \hat{u}$ everywhere.

Conversely, Theorem 3.3 and the representation formula (2.1) imply

$$u(x,t) = g(y) + Dg(y)\cdot(x-y) - tH(Dg(y)) = g(y) + Dg(y)\cdot(x-y) - t\hat{H}(Dg(y)),$$

the point $y = y(x,t)$ given by (3.12). Since $\frac{x-y}{t} = D\hat{H}(Dg(y))$, we therefore have

$$u(x,t) = g(y) + t\hat{L}(\tfrac{x-y}{t}) \geq \inf_{z\in\mathbb{R}^n}\left\{g(z) + t\hat{L}(\tfrac{x-z}{t})\right\} = \hat{u}(x,t),$$

according to the Hopf–Law formula (1.6). □

The set $\boldsymbol{\Sigma_2 \subseteq \Sigma}$. Next, we define

$$\Sigma_2 := \Sigma \cap \mathrm{co}(\Sigma - \Sigma_1). \tag{3.15}$$

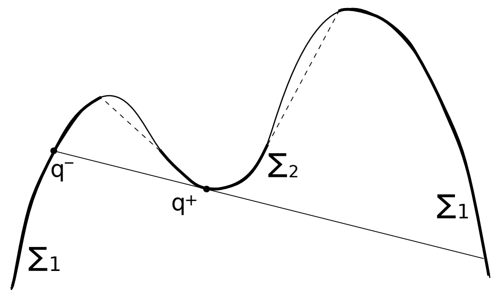

Figure 3: The sets Σ_1 and Σ_2

The important property is that if $q^+ = \nabla u(x,t)$ belongs to a smooth part of Σ_2, then *the tangent plane at q^+ intersects Σ only at other points q^- that belong to* Σ_1: see Figure 3. We henceforth assume that all such points q^- in fact lie within uniformly convex portions of Σ_1.

The geometric consequence in physical space is that a straight backwards characteristic starting at (x,t) either (i) will extend all the way back to the starting time $t = 0$, or else (ii) will tangentially hit an envelope shock at the point (x_1, t_1) with $0 < t_1 < t$. In the latter case, the state $q^- = \nabla u^-(x_1, t_1)$ belongs to the set Σ_1 and consequently, the backward characteristic from that point must extend all the way to $t = 0$, according to Theorem (3.3). Thus, *if $\nabla u(x,t) \in \Sigma_2$, then there exists a piecewise-linear backwards characteristic extending to $t = 0$ with at most two pieces.*

We can in principle extend the foregoing analysis to identify subsets $\Sigma_k \subset \Sigma$ for $k = 2, \ldots,$ with the property that if $\nabla u(x,t) \in \Sigma_k$, then there exists a piecewise-linear backwards characteristic extending to $t = 0$ with at most k pieces.

4 Formulas for equivocal curves and surfaces

A. Melikyan's book [M] introduces generalized characteristics for Hamilton–Jacobi type PDE and in particular uses solutions of these ODE to deduce information about the properties of what this paper calls equivocal surfaces. We present now some new and simple derivations, with attention paid to the geometric and analytic properties of equivocal curves ($n = 1$) and surfaces ($n \geq 2$).

4.1 One space dimension. Consider first the case $n = 1$.

$$\text{(4.1)} \qquad \begin{cases} u_t + H(u_x) = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ \qquad\qquad u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

which case is equivalent to studying scalar conservation laws: see for instance Ballou [Ba1, Ba2], Dafermos [D].

We assume that the curve $\Gamma = \{x = s(t)\}$ is an equivocal curve, as drawn in Figure 1. We therefore have the ODE

$$\dot{s}(t) = H'(u_x^+(s(t),t)), \tag{4.2}$$

where we denote by $\nabla u^+ = \nabla u^+(s(t),t)$ and $\nabla u^- = \nabla u^-(s(t),t)$ the right- and left-hand values of the gradient $\nabla u = (u_x, u_t)$ along the curve at time t. Figure 4 illustrates typical locations of $q^\pm := \nabla u^\pm$, and the line segment connecting $q+$ and q^- is tangent to the curve Σ at q^+, since we have a contact discontinuity there. We think of u^- as providing "incoming" information to Γ, whereas u^+ is "outgoing". For definiteness, we assume that locally $u_x^- < u_x^+$, as drawn.

Blow up of u_{xx}^+ along Γ. A goal of this section is to augment (4.2) by computing as well $\ddot{s}$. This does not follow from just naively differentiating (4.2), since the geometric form of Γ as an envelope of the outgoing characteristics (see Figure 1) forces u_{xx}^+ to be unbounded near Γ, as we demonstrate next.

Given a point $(s(t),t) \in \Gamma$ and a small time $\tau > t$, let

$$r(\tau) := s(t) + (\tau - t)H'(u_x^+(s(t),t));$$

so that $(r(\tau),\tau)$ lies on the tangent line to Γ passing through $(s(t),t)$.

LEMMA 4.1 *Assume $\ddot{s}(t) \neq 0$. Then*

$$u_{xx}(r(\tau),\tau) = \frac{1}{(\tau - t)H''(u_x^+(s(t),t))}; \tag{4.3}$$

and therefore if $H''(u_x^+(s(t),t)) \neq 0$, we have $|u_{xx}(r(\tau),\tau)| \to \infty$ as $\tau \to t^+$.

Proof. Select a small $h > 0$ and define

$$r(\tau,h) := s(t+h) + (\tau - t - h)H'(u_x^+(s(t+h),t+h)).$$

Then $(r(\tau,h),\tau)$ lies on the tangent line to Γ passing through $(s(t+h),t+h)$ and so

$$r(\tau,h) = s(t+h) + (\tau - t - h)H'(u_x(r(\tau,h),\tau)),$$

since ∇u is constant along this tangent line. Differentiate in h and then set $h = 0$:

$$r_h(\tau,0) = \dot{s}(t) - H'(u_x^+(s(t),t)) + (\tau - t)H''(u_x(r(\tau),\tau))u_{xx}(r(\tau),\tau)r_h(\tau,0).$$

Then (4.2) implies $r_h(\tau,0)((\tau - t)H''(u_x^+(s(t),t))u_{xx}(r(\tau),\tau) - 1) = 0$. Since $\ddot{s}(t) \neq 0$ implies $r_h(\tau,0) \neq 0$, we obtain (4.3). ∎

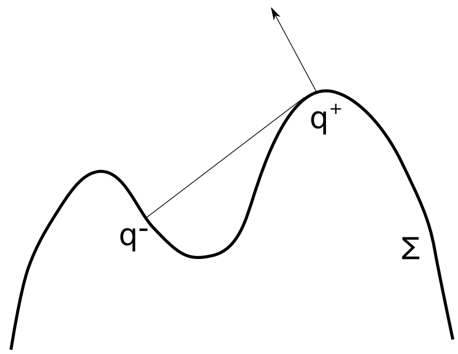

Figure 4: Left and right states on Σ

We remark that the blow up of u_{xx}^+ is also obvious since the initial-value problem for ODE (4.2), starting at a point on Γ, has two distinct solutions, one tracing out the equivocal curve and the other following the straight characteristic leaving Γ tangentially.

Computing $\ddot{s}$. Since we cannot just differentiate (4.2), we must proceed indirectly.

THEOREM 4.2 (i) *We have*

$$\frac{d}{dt}\nabla u^+(s(t),t) = \gamma(t)(\nabla u^+(s(t),t) - \nabla u^-(s(t),t)) \tag{4.4}$$

for

$$\gamma = \frac{u_{xx}^-(H'(u_x^+) - H'(u_x^-))^2}{H''(u_x^+)(u_x^+ - u_x^-)^2}, \tag{4.5}$$

$u_x^\pm$ *and* u_{xx}^- *evaluated at* $(s(t),t)$.

(ii) *Furthermore,*

$$\ddot{s} = \frac{u_{xx}^-(H'(u_x^+) - H'(u_x^-))^2}{u_x^+ - u_x^-}, \tag{4.6}$$

$u_x^\pm$ *and* u_{xx}^- *evaluated at* $(s(t),t)$.

Since $u_x^+ > u_x^-$ and $\ddot{s} \leq 0$, it follows that $u_{xx}^- \leq 0$ along Γ, as drawn in Figure 1. This is consistent with the comment following Theorem 3.2 that equivocal surfaces cannot arise for convex initial data. See also Theorem 4.4.

Proof. 1. Define $K(q) = p_2 + H(p_1)$ for $q = (p_1, p_2)$; so that $\Sigma = \{K = 0\}$ and ∇K is perpendicular to Σ. Then

$$\nabla K(\nabla u^+(s(t),t)) \cdot (\nabla u^+(s(t),t) - \nabla u^-(s(t),t)) = 0, \tag{4.7}$$

since we have a contact discontinuity. Differentiating the identity $K(\nabla u^+(s(t),t)) = 0$, we see furthermore that

$$\nabla K(\nabla u^+(s(t),t)) \cdot \frac{d}{dt}\nabla u^+(s(t),t) = 0.$$

In view of the contact discontinuity identity (4.7) and since we are in the two-dimensional (x,t) plane, it follows that $\frac{d}{dt}\nabla u^+$ is parallel to $\nabla u^+ - \nabla u^-$.

2. Observe next that locally along Γ, ∇u^- and ∇u^+ are functionally related, depending only upon the geometry of the surface Σ. We there can locally define a function ψ that maps each point u_x^- to the corresponding u_x^+. Then

$$\psi(u_x^-) = u_x^+ \tag{4.8}$$

along Γ.

Since the line segment connecting $(p, -H(p)) \in \Sigma$ to $(\psi(p), -H(\psi(p))) \in \Sigma$ is tangent to Σ at the latter point, we have $(\psi(p) - p)H'(\psi(p)) = H(\psi(p)) - H(p)$. Differentiate:

$$H''(\psi(p))\psi'(p) = \frac{H'(\psi(p)) - H'(p)}{\psi(p) - p}. \tag{4.9}$$

Now compute

$$\begin{aligned} \ddot{s} &= \frac{d}{dt}H'(u_x^+) = \frac{d}{dt}H'(\psi(u_x^-)) = H''(\psi(u_x^-))\psi'(u_x^-)(u_{xx}^-\dot{s} + u_{xt}^-) \\ &= \frac{H'(u_x^+) - H'(u_x^-)}{u_x^+ - u_x^-}(u_{xx}^-\dot{s} + u_{xt}^-) = \frac{H'(u_x^+) - H'(u_x^-)}{u_x^+ - u_x^-}u_{xx}^-(H'(u_x^+) - H'(u_x^-)). \end{aligned}$$

This proves (4.6).

3. Differentiating $\dot{s} = H'(u_x^+)$ and using the first component of the vector equation (4.4) gives $\ddot{s} = \gamma H''(u_x^+)(u_x^+ - u_x^-)$. This and (4.6) give (4.5). □

4.2 Higher dimensions. We move now to $n \geq 2$. The Hamilton–Jacobi PDE is therefore

$$u_t + H(Du) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty); \tag{4.10}$$

and as before we write $\nabla u = (Du, u_t), Du = D_x u$.

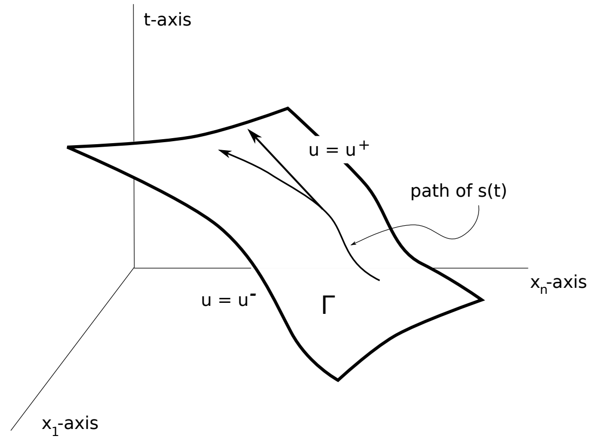

Figure 5: Path of a singular characteristic

Hereafter we let Γ denote an equivocal surface, which we think of as a smooth hypersurface locally parameterized as a graph in the time direction: $\Gamma = \{(x,t) \in \mathbb{R}^n \times (0,\infty) \mid t = \sigma(x)\}$. We suppose that u is continuous across Γ, but its gradient is discontinuous. Denote by ∇u^- and ∇u^+ the values of the gradient from below and from above. Then $\nabla u^+ - \nabla u^-$ is normal to Γ.

For this higher dimensional ($n \geq 2$) problem, we adopt a fundamentally different viewpoint from that above. We now regard the values of the solution u^- on the "incoming" side to be known and ask how to connect the lower states ∇u^- to upper states ∇u^+ along Γ. This is not an issue for $n = 1$, at least locally when the geometry is as sketched in Figure 4, since given $q^- = \nabla u^-$ there is locally a unique choice of $q^+ = \nabla u^+$. This is not so however for $n \geq 2$.

Inspired by Melikyan [M], we instead introduce the *singular characteristic equations*

$$\begin{cases} \dot{\mathbf{s}}(t) = DH(\mathbf{p}(t)) \\ \dot{\mathbf{p}}(t) = \gamma(t)(\mathbf{p}(t) - Du^-(\mathbf{s}(t),t)), \end{cases} \tag{4.11}$$

where

$$\gamma = \frac{(DH(\mathbf{p}) - DH(Du^-))^T D^2u^-(DH(\mathbf{p}) - DH(Du^-))}{(\mathbf{p} - Du^-)^T D^2H(\mathbf{p})(\mathbf{p} - Du^-)}, \tag{4.12}$$

Du^- and D^2u^- evaluated at $(\mathbf{s}(t),t)$.

Given a point $(x_0,t_0) \in \Gamma$, we assume we can at least for small times $t \geq t_0$ solve the ODE (4.11), (4.12) with the initial conditions

$$\mathbf{s}(t_0) = x_0, \;\; \mathbf{p}(t_0) = p_0, \tag{4.13}$$

the vector $p_0 \in \mathbb{R}^n$ selected to satisfy

$$(p_0 - Du^-(x_0,t_0)) \cdot DH(p_0) = H(p_0) - H(Du^-(x_0,t_0)) \tag{4.14}$$

and

$$p_0 \neq Du^-(x_0,t_0)),\ (p_0 - Du^-(x_0,t_0))^T D^2H(p_0)(p_0 - Du^-(x_0,t_0)) \neq 0. \tag{4.15}$$

We further assume that the equivocal surface Γ to be made up of trajectories of the vector ODE: see Figure 5. The idea is that by solving the singular characteristic ODE we can compute $\mathbf{p}$, which is a candidate for $Du^+(\mathbf{s},t)$. However there are many subtleties here: see the comments after the proof of the following assertion.

THEOREM 4.3 *We have*

$$(\mathbf{p}(t) - Du^-(\mathbf{s}(t),t)) \cdot DH(\mathbf{p}(t)) = H(\mathbf{p}(t)) - H(Du^-(\mathbf{s}(t),t)) \tag{4.16}$$

for times $t \geq t_0$ *for which* (4.11), (4.12) *has a solution.*

Proof. 1. It is convenient to calculate in $n+1$ variables; and so we introduce the functions

$$\mathbf{r}(t) := (\mathbf{s}(t),t),\ \mathbf{q}(t) := (\mathbf{p}(t), q_{n+1}(t)).$$

As before we write $K(q) = p_{n+1} + H(p)$ for $q = (p, p_{n+1})$. Then $\nabla K(q) = (DH(p),1)$; and consequently the first equation in (4.11) becomes

$$\dot{\mathbf{r}} = \nabla K(\mathbf{q}). \tag{4.17}$$

We also extend the second ODE to (4.11) to read

$$\dot{\mathbf{q}} = \gamma(\mathbf{q} - \nabla u^-(\mathbf{r})). \tag{4.18}$$

For the initial condition we take $\mathbf{q}(t_0) = q_0 := (p_0, -H(p_0))$. Then $K(q_0) = 0$ and

$$\begin{aligned} &(q_0 - \nabla u^-(x_0,t_0)) \cdot \nabla K(q_0) \\ &\qquad = (p_0 - Du^-(x_0,t_0)) \cdot DH(p_0) - H(p_0) + H(Du^-(x_0,t_0)) = 0 \end{aligned} \tag{4.19}$$

according to (4.16), since $u_t^- + H(Du^-) = 0$.

2. Introduce finally the expressions

$$\begin{cases} a(t) := K(\mathbf{q}(t)) \\ b(t) := (\mathbf{q}(t) - \nabla u^-(\mathbf{r}(t))) \cdot \nabla K(\mathbf{q}(t)). \end{cases}$$

We will show that both of these functions vanish identically; this implies (4.16).

Firstly, owing to (4.18) we have

$$\dot{a} = \gamma \nabla K(\mathbf{q}) \cdot (\mathbf{q} - \nabla u^-(\mathbf{r})) = \gamma b. \tag{4.20}$$

Secondly,

$$\begin{aligned}
\dot{b} &= (\dot{\mathbf{q}} - \nabla^2 u^-(\mathbf{r})\dot{\mathbf{r}}) \cdot \nabla K(\mathbf{q}) + (\mathbf{q} - \nabla u^-(\mathbf{r})) \cdot \nabla^2 K(\mathbf{q})\dot{\mathbf{q}} \\
&= \dot{a} - \nabla K(\mathbf{q})^T \nabla^2 u^-(\mathbf{r}) \nabla K(\mathbf{q}) \\
&\qquad + \gamma(\mathbf{q} - \nabla u^-(\mathbf{r}))^T \nabla^2 K(\mathbf{q})(\mathbf{q} - \nabla u^-(\mathbf{r})) \\
&= \dot{a},
\end{aligned}$$

provided

$$\gamma = \frac{\nabla K(\mathbf{q})^T \nabla^2 u^-(\mathbf{r}) \nabla K(\mathbf{q})}{(\mathbf{q} - \nabla u^-(\mathbf{r}))^T \nabla^2 K(\mathbf{q})(\mathbf{q} - \nabla u^-(\mathbf{r}))}. \tag{4.21}$$

Assuming this for the moment, we see that $\dot{b} = \dot{a} = \gamma b$, according to (4.20). So $b \equiv 0$ since (4.19) implies $b(t_0) = 0$. Then $a \equiv 0$, as $a(t_0) = 0$.

3. It remains to show that (4.21) agrees with (4.12). Now $\nabla K = (DH, 1)$ and

$$\nabla^2 K = \begin{pmatrix} D^2 H & 0 \\ 0 & 0 \end{pmatrix}, \ \nabla^2 u^- = \begin{pmatrix} D^2 u^- & Du_t^- \\ Du_t^- & u_{tt}^- \end{pmatrix}.$$

Therefore

$$(\mathbf{q} - \nabla u^-(\mathbf{r}))^T \nabla^2 K(\mathbf{q})(\mathbf{q} - \nabla u^-(\mathbf{r})) = (\mathbf{p} - Du^-(\mathbf{r}))^T D^2 H(\mathbf{p})(\mathbf{p} - Du^-(\mathbf{r})).$$

In addition, $u_t^- + H(Du^-) = 0$, and consequently

$$u_{tt}^- + DH(Du^-)Du_t^- = 0, \quad Du_t^- + DH(Du^-)D^2 u^- = 0.$$

Thus

$$\begin{aligned}
\nabla K(\mathbf{q})^T \nabla^2 u^- \nabla K(\mathbf{q}) &= DH(\mathbf{p})^T D^2 u^- DH(\mathbf{p}) + 2Du_t^- DH(\mathbf{p}) + u_{tt}^- \\
&= (DH(\mathbf{p}) - DH(Du^-))^T D^2 u^- (DH(\mathbf{p}) - DH(Du^-)).
\end{aligned}$$

These identities give (4.12). □

Remarks. (i) The system of ODE (4.11), (4.12) are a special case of (1.105) in Melikyan [M], with simpler notation. It follows that

$$\ddot{\mathbf{s}} = \gamma D^2 H(Du^+)(Du^+ - Du^-), \tag{4.22}$$

where

$$Du^+(\mathbf{s}(t), t) = \mathbf{p}(t), \tag{4.23}$$

in accord with (4.6).

(ii) An observation of P. Bernhard [B], recounted in Lewin [L, Lemma 10.6.2], is that for the value $u = u(x)$ of a two-person, zero sum differential game we have

$$\frac{d}{dt} Du^+(\mathbf{s}) = \gamma(Du^+(\mathbf{s}) - Du^-(\mathbf{s})) \tag{4.24}$$

for some scalar function γ, when there trajectory of the curve $\mathbf{s}$ solving the ODE $\dot{\mathbf{s}} = DH(Du^+(\mathbf{s}))$ lies along an equivocal surface. This is consistent with the singular characteristic equations (4.11), with the difference that in (4.24) the outgoing state Du^+ is assumed known. The derivation seems to require that u^+ be C^2 all the way up to the equivocal surface Γ, and this is perhaps an issue in light of Lemma 4.1 above. □

We can in addition determine the signs of the terms in the expression (4.12) for γ:

THEOREM 4.4 *Under the identification* (4.23)*, we have*

$$(Du^+ - Du^-)^T D^2H(Du^+)(Du^+ - Du^-) \geq 0 \tag{4.25}$$

and

$$(DH(Du^+) - DH(Du^-))^T D^2u^-(DH(Du^+) - DH(Du^-)) \leq 0, \tag{4.26}$$

$Du^\pm$ *and* D^2u^- *evaluated at* $(\mathbf{s}(t), t)$*.*

These are special cases of Melikyan [M, Lemma 2.3]. The second inequality (4.26) follows from Melikyan's [M, Theorem 1.6] on the solvability of irregular, noncharacteristic problems for nonlinear first-order PDE.

The geometric meaning of these inequalities is fairly clear. Indeed, (4.25) holds since the line segment $[Du^-, Du^+]$ lies above the surface Σ, which is the graph of $-H$, and is tangent to the surface at Du^+. According to (4.22) and (4.12), inequality (4.26) records the fact that in Figure 5 the inner product of $\ddot{\mathbf{s}}$ with the normal vector $Du^+ - Du^-$ is nonpositive.

5 Barriers

This section does not fit within our overall theme of envelope constructions, but instead provides some new observations about barriers (= surfaces of discontinuity) for viscosity

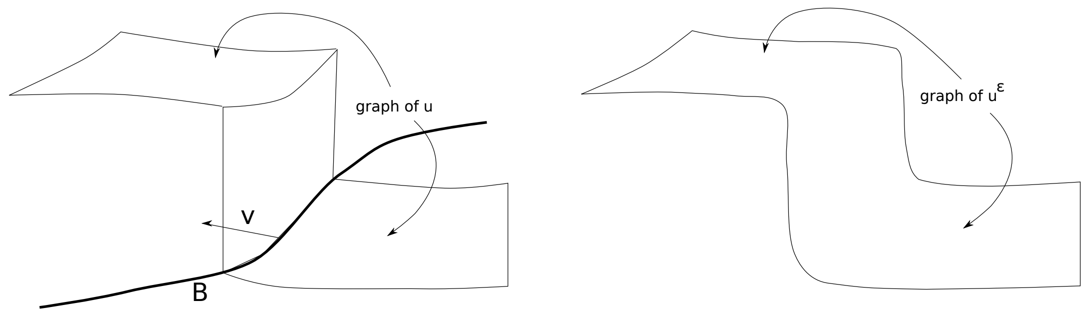

Figure 6: Barrier and smooth approximation

solutions of Hamilton–Jacobi type PDE. We will refer back to these insights in the subsequent discussion of differential games and HJI equations.

We assume for this section that $u^\varepsilon = u^\varepsilon(x)$ solves

$$-\epsilon \Delta u^\epsilon + H(Du^\epsilon, x) = 0 \quad \text{in } U. \tag{5.1}$$

We assume hereafter that u^ϵ converges uniformly to u outside each open neighborhood of the smooth barrier hypersurface B. Suppose also that u is smooth on each side of B, but is discontinuous across B. Finally, let ν denote the unit normal to B pointing towards the side where u is larger: see Figure 6. Write u^+ to denote u in the region into which ν points, and u^- for u in the region from which ν points. We suppose that u^+ and u^- are smooth from each side, up to the barrier B. Lastly, we will suppose we can locally smoothly extend $u^\pm$ beyond Γ and assume that

$$u^\epsilon \leq u^+ + o(1), u^\epsilon \geq u^- - o(1) \tag{5.2}$$

near zero.

The *recession function* associated with a Hamiltonian $H = H(p,x)$ is

$$K(p,x) := \lim_{\lambda \to \infty} \frac{H(q + \lambda p, x)}{\lambda}, \tag{5.3}$$

provided this limit exists uniformly for p, q in compact sets.

THEOREM 5.1 *Let $x \in B$. Then under the foregoing conditions, we have*

(i)

$$\begin{cases} H(Du^+ + \lambda\nu, x) \leq 0 \\ H(Du^- + \lambda\nu, x) \geq 0 \end{cases} \quad \textit{for all } \lambda \geq 0. \tag{5.4}$$

(ii) *In particular,*

$$D_pH(Du^+, x)\cdot\nu \le 0,\ \ D_pH(Du^-, x)\cdot\nu \ge 0; \tag{5.5}$$

and if the recession function K exists,

$$K(\nu, x) = 0. \tag{5.6}$$

While the inequalities (5.4) are a rigorous deduction, it would be difficult to verify the hypotheses about the behavior of the functions u^ϵ near B. It is therefore probably best in practice to regard (5.4) as heuristics.

The identity (5.6) for differential games is Isaacs' observation that a "barrier is a semipermeable surface": see [I, page 204]. The inequalities (5.4) seem to be new and provide more geometric information. The idea of the proof that since u is discontinuous along the surface B and the smooth functions u^ϵ converge nicely to u, we can touch the graph of u^ϵ near B from above and below by smooth functions that tilt strongly in the ν direction.

See Rapaport [R] for more about barriers from the viewpoint of viscosity solutions.

Proof. 1. We may assume without loss that $x = 0$ and $\nu = e_n$; consequently the barrier surface near 0 can be written as the graph $\{x_n = \gamma(x')\}$ of a smooth function γ of the variables $x' = (x_1, \dots, x_{n-1})$. We for definiteness take $u^+(0) = 0, u^-(0) = -1$. Redefining H if necessary, we can suppose also that $Du^+(0) = 0$. Then near 0 we have

$$\begin{cases} u^+(x) = O(|x|^2) & \text{for } x_n > \gamma(x') \\ u^-(x) = -1 + a\cdot x + O(|x|^2) & \text{for } x_n < \gamma(x'), \end{cases} \tag{5.7}$$

where $a := Du^-(0)$.

Fix a small number $r > 0$ and then put $s := C_1 r^2$, the constant C_1 selected so large that

$$\begin{cases} x \in B(0,r)\cap\{x_n \ge s\} \ \text{ implies } x_n > \gamma(x') \\ x \in B(0,r)\cap\{x_n \le -s\} \ \text{ implies } x_n < \gamma(x'). \end{cases}$$

Fix any $\lambda > 0$, and introduce then the comparison function

$$\phi(x) = \lambda x_n + \alpha|x|^2,$$

the constant $\alpha > 0$ to be adjusted later.

2. We claim that for sufficiently small $\epsilon > 0$,

$$u^\epsilon - \phi \text{ attains its maximum at a point } x_\epsilon \text{ in the interior of } B(0,r). \tag{5.8}$$

To see this, we observe from (5.7) that

$$\begin{aligned}(u^\epsilon - \phi)(2se_n) &= (u^+ - \phi)(2se_n) + (u^\epsilon - u^+)(2se_n)\\ &\geq -O(s^2) - 2\lambda s - 4\alpha s^2 - o(1)\\ &= -2\lambda C_1 r^2 - O(r^4) - o(1)\\ &=: \gamma^\epsilon,\end{aligned}$$

where $o(1) = |(u^\epsilon - u^+)(2se_n)| \to 0$ as $\epsilon \to 0$, with $r > 0$ fixed.

Now (5.7) implies

$$\max_{\partial B(0,r)\cap\{x_n \geq s\}} (u^\epsilon - \phi) \leq C_2 r^2 - \alpha r^2 + o(1) < \gamma^\epsilon,$$

provided we select $\alpha > 3\lambda C_1 + C_2$ and take first r and then ϵ sufficiently small. In this formula we write $o(1) = \max_{\partial B(0,r)\cap\{x_n\geq s\}} |u^\epsilon - u^+| \to 0$ as $\epsilon \to 0$, for each r. The second line of (5.7) implies

$$\max_{\partial B(0,r)\cap\{x_n \leq -s\}} (u^\epsilon - \phi) \leq -1 + O(r) + o(1) < \gamma^\epsilon,$$

again for small r and ϵ. Finally, we see from (5.2) that

$$\max_{\partial B(0,r)\cap\{-s\leq x_n \leq s\}} (u^\epsilon - \phi) \leq o(1) + C_2 r^2 + \lambda C_1 r^2 - \alpha r^2 < \gamma^\epsilon,$$

again provided $\alpha > 3\lambda C_1 + C_2$ and r, ϵ are small. The foregoing calculations show that $(u^\epsilon - \phi)(2se_n) > \max_{\partial B(0,r)}(u^\epsilon - \phi)$ and so the assertion (5.8) is proved.

3. Owing to (5.8) and the PDE (5.1), we have

$$-\epsilon\Delta\phi(x_\epsilon) + H(D\phi(x_\epsilon), x_\epsilon) \leq 0.$$

Thus $-2\epsilon + H(\lambda e_n + 2\alpha x_\epsilon, x_\epsilon) \leq 0$. Let $\epsilon \to 0$ and then $r \to 0$, to deduce

$$H(\lambda e_n, 0) \leq 0.$$

This is the first inequality in (5.4), since $x = 0, \nu = e_n, Du^+(0) = 0$. The second inequality follows similarly upon our touching the graph of u^ϵ from below by a smooth function.

4. Since $H(D^+u(x), x) = 0$, (5.4) implies

$$D_pH(Du^+, x)\cdot\nu = \lim_{\lambda\to 0+} \frac{H(Du^+ + \lambda\nu, x) - H(Du^+, x)}{\lambda} \leq 0;$$

and the other inequality in (5.5) is similar. Owing to (5.4) and (5.3), we have

$$\begin{cases} K(\nu, x) = \lim_{\lambda\to\infty} \frac{H(Du^+ + \lambda\nu, x)}{\lambda} \leq 0,\\ K(\nu, x) = \lim_{\lambda\to\infty} \frac{H(Du^- + \lambda\nu, x)}{\lambda} \geq 0.\end{cases}$$

□

6 Differential games

The books of Isaacs [I], Friedman [F] and Lewin [L] on two-person, zero-sum differential games (see also Cardaliaguet [C] and Basar–Olsder [B-O]) provide many truly fascinating examples illustrating the complexity of the singular structure of solutions to nonlinear first-order PDE. The concrete problems in these books however present many mathematical difficulties, the origins of which we can roughly classify as follows: (i) the Hamiltonians $H = H(p, x)$ are always nonconvex in p; (ii) the Hamiltonians are mostly not C^1; (iii) the Hamiltonians are often not coercive (that is, the algebraic identity $H(p, x) = 0$ does not imply p is bounded); and (iv) the PDE often involve unclear or unspecified boundary conditions. This section revisits some of these issues, in light of the foregoing theory.

6.1 Notation for differential games. We introduce now the rudiments of the viscosity solution interpretation of two-person zero-sum differential game theory: see Cardaliaguet [C] or [E-S] for more explanation. Let us consider the dynamics

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \alpha, \beta) & (0 \le t \le \tau) \\ \mathbf{x}(0) = x, \end{cases} \tag{6.1}$$

where $\alpha : [0, \tau] \to A$ is the control for the maximizing player and $\beta : [0, \tau] \to B$ is the control for the minimizing player. The associated payoff is

$$\int_0^\tau r(\mathbf{x}, \alpha, \beta)\, dt + g(\mathbf{x}(\tau)). \tag{6.2}$$

We hereafter assume the *minimax condition* that

$$\min_{b \in B} \max_{a \in A} \{\mathbf{f}(x, a, b) \cdot p + r(x, a, b)\} = \max_{a \in A} \min_{b \in B} \{\mathbf{f}(x, a, b) \cdot p + r(x, a, b)\} \tag{6.3}$$

for all x, p.

Let u denote the corresponding *value function*, which, if continuous, solves the *Hamilton–Jacobi–Isaacs* (HJI) equation

$$H(Du, x) = 0 \tag{6.4}$$

in the viscosity sense, for the *game theory Hamiltonian*

$$H(p, x) = -\min_{b \in B} \max_{a \in A} \{\mathbf{f}(x, a, b) \cdot p + r(x, a, b)\}. \tag{6.5}$$

The minus sign here is to make the PDE (6.4) consistent with standard viscosity solution interpretations. See [E-S] for precise definitions and explanations of all this.

6.2 Equivocal surfaces. In this section we discuss the connections between the measures $\gamma_{x,t}$ introduced in Section 2 and the advent of equivocal surfaces. In PDE terms, we see that if the measure $\gamma_{x,t}$ is not a point mass, our viscosity solution u has "envelope shocks", meaning discontinuity surfaces for the gradient ∇u from which characteristics leave tangentially going forward in time.

No running payoffs. For differential games with no running payoffs, the Hamiltonian

$$H(p,x) = \max_{a\in A}\min_{b\in B}\{\mathbf{f}(a,b,x)\cdot p\}$$

is positively homogeneous of degree one, and thus $D_pH(p,x)\cdot p = H(p,x)$ for $p \neq 0$. Theorem (3.1) therefore applies and we see that the support of the measure $\gamma_{x,t}$ lies in the set $\{g = u(x,t)\}$.

The game theoretic interpretation is that a homogeneous H corresponds to a game with no running payoff. If the support of $\gamma_{x,t}$ is not a single point, there can be many optimal trajectories, but they all end up at points y where the payoff $g(y)$ is the same and equals $u(x,t)$.

Including running payoffs. For differential games with running payoffs, the Hamiltonian (6.5) is not homogeneous. But we can as follows add another variable, to reduce to the previous case. For this, define

$$\begin{cases} \tilde{x} = (x, x_{n+1}), \tilde{u}(\tilde{x},t) = u(x,t) + x_{n+1} \\ \tilde{p} = (p, p_{n+1}), \tilde{H}(\tilde{p}) = p_{n+1}H(p/p_{n+1}). \end{cases}$$

Then $\tilde{H}$ is positively homogeneous and $\tilde{u}$ is the unique viscosity solution of

$$\text{(6.6)} \qquad \begin{cases} \tilde{u}_t + \tilde{H}(D\tilde{u}) = 0 & \text{in } \mathbb{R}^{n+1}\times(0,\infty) \\ \tilde{u} = \tilde{g} & \text{on } \mathbb{R}^{n+1}\times\{t=0\}, \end{cases}$$

for $\tilde{g}(\tilde{y}) = g(y) + y_{n+1}$.

We introduce also the analog of (2.5):

$$-\tilde{\sigma}^\varepsilon_t - \operatorname{div}(\tilde{\sigma}^\varepsilon D\tilde{H}(D\tilde{u}^\varepsilon)) = \varepsilon\Delta\tilde{\sigma}^\varepsilon.$$

In view of the definition of $\tilde{H}$ this PDE reads

$$\text{(6.7)} \qquad -\tilde{\sigma}^\varepsilon_t - \operatorname{div}_x(\tilde{\sigma}^\varepsilon DH(Du^\varepsilon)) - (\tilde{\sigma}^\varepsilon(H(Du^\varepsilon) - DH(Du^\varepsilon)\cdot Du^\varepsilon))_{x_{n+1}} = \varepsilon\Delta\tilde{\sigma}^\varepsilon.$$

Then as above we have:

THEOREM 6.1 (i) *For $\mathcal{L}^{n+2}$ almost every point $(\tilde{x}, t) \in \mathbb{R}^{n+1} \times (0, \infty)$,*

$$\tilde{g}(\tilde{y}) = \tilde{u}(\tilde{x}, t) \quad \text{for } \tilde{\gamma}_{\tilde{x},t} \text{ almost every } \tilde{y} \in \mathbb{R}^{n+1}. \tag{6.8}$$

(ii) *In particular, for $\mathcal{L}^{n+1}$ almost every $(x, t) \in \mathbb{R}^n \times (0, \infty)$, we have*

$$u(x, t) = g(y) + y_{n+1} \quad \text{for } \tilde{\gamma}_{x,t} \text{ almost every } \tilde{y} \in \mathbb{R}^{n+1}. \tag{6.9}$$

A largely open question is how better to interpret (6.9) in light of differential game theory, and in particular to understand y_{n+1} as recording the total running cost of an optimal trajectory ending up at the point y. If everything were smooth, we should expect from the classical theory of characteristics that

$$u(x, t) = g(y) + \int_0^t DH(Du(\mathbf{x}(s), s)) \cdot Du(\mathbf{x}(s), s) - H(Du(\mathbf{x}(s), s))\, ds, \tag{6.10}$$

where $\{(\mathbf{x}(s), s) \mid 0 \leq s \leq t\}$ parameterizes the optimal trajectory, starting at x at time t. However (6.10) does not in general have an obvious meaning since we expect there to be many optimal paths, which are straight lines in regions where u is smooth, but can bend at and along the equivocal surfaces.

6.3 An example of Isaacs. We conclude with a look at a particular Hamilton-Jacobi-Isaacs PDE, which is simple looking but nevertheless suffers from all of the difficulties (i)-(iv) listed at the start of Section 6. This example illustrates several of the ideas discussed before, in particular the theory for equivocal curves and barriers.

Isaacs in [I, Example 8.4.3] introduces the (unnamed) differential game with the dynamics

$$\begin{cases} \dot{x} = c(y) + \cos\alpha \\ \dot{y} = 2\beta + \sin\alpha, \end{cases} \tag{6.11}$$

where $0 \leq \alpha \leq 2\pi$ and $-1 \leq \beta \leq 1$. Here c is a given positive and nondecreasing function of y. The game is played in the upper half plane $U := \mathbb{R}^2 \cap \{y \geq 0\}$ and the payoff is the time τ until the trajectory exits U along the positive x-axis. Hence the running payoff is $r \equiv 1$. (See also Basar–Olsder [B-O, page 466].)

HJI equation, barriers, equivocal curve. The corresponding Hamiltonian is

$$H(p, y) = -c(y)p_1 - |p| + 2|p_2| - 1; \tag{6.12}$$

and the HJI equation for the value function $u = u(x, y)$, the time to exit given we start at (x, y), reads

$$H(Du, y) = -c(y)u_x - |Du| + 2|u_y| - 1 = 0 \quad \text{in } U \tag{6.13}$$

in the viscosity sense. We clearly have the boundary condition $u = 0$ on $\{x \geq 0, y = 0\}$; but it is unclear what boundary conditions should be required along $\{x < 0, y = 0\}$.

Isaacs in [I, p 295] provides a sketch for optimal paths, here roughly redrawn as Figure 7, where B a barrier and ES is an equivocal curve, across which optimal strategies change. We draw the optimal game theoretic paths as moving in the opposite direction from the conventional PDE characteristics. We have $u = u^- = y$ to the right of $B \cup ES$.

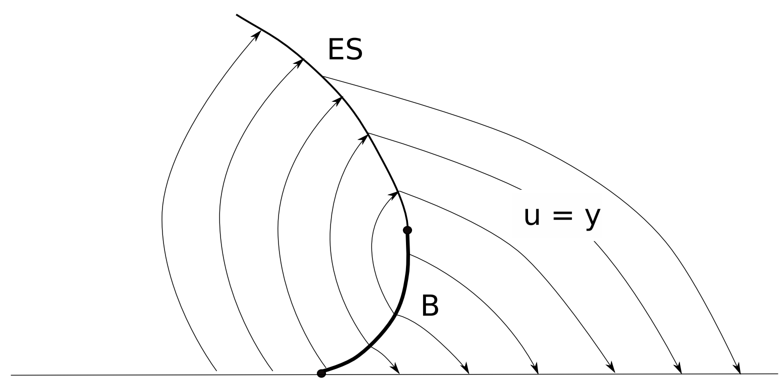

Figure 7: Optimal paths, a barrier and an equivocal curve

We provide next a geometric interpretation and justification of this picture, by extracting information about the left states ∇u^+ and right state $\nabla u^- = e_2$ along ES from the geometry of the level sets of H for different values of the parameter $c = c(y)$. Let us for simplicity assume that $0 < c(y) < 1$ for $0 \leq y < 1$ and $c(y) > 1$ for $y \geq 1$.

Observe from Figure 8 that for values of $c = c(y) < 1$ the level curves in the $q = (p_1, p_2)$ plane comprise two distinct pieces, each of which is half of a different hyperbola (tilted at different angles). For $c = 1$ a asymptote of both hyperbolas is the negative real axis. A geometric consequence is that for $c \leq 1$ we cannot connect the state $q^- = \nabla u^- = e_2$, which lies on the upper part of the characteristic curve $\Sigma = \{H = 0\}$, to any state q^+ lying on the lower part. For if we could, the line segment $[q^-, q^+]$ connecting the two states would have to be tangent to Σ at q^+; and this is not possible geometrically. Hence there must be a barrier B existing for $0 \leq y \leq 1$.

The unit normal $\nu = (\nu^1, \nu^2)$ to the barrier is given by Isaacs' condition (5.6):

$$0 = K(\nu, y) = -c(y)\nu^1 - 1 + 2|\nu^2|.$$

Notice also from the picture for $c = .5$ that $H(\nabla u^- + \lambda\nu, y) = H(e_2 + \lambda\nu, y) \geq 0$ for all $\lambda \geq 0$, as predicted by the barrier inequalities (5.4).

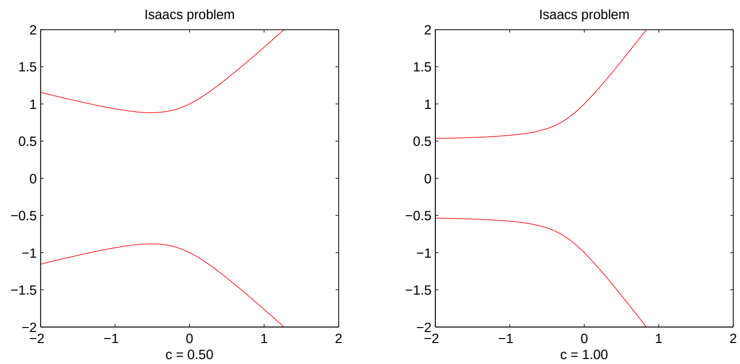

Figure 8: Some characteristic curves $H = 0$ for $c \leq 1$

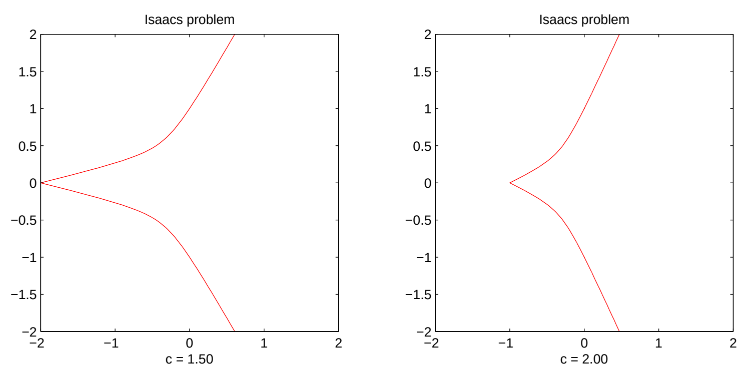

Figure 9: Some characteristic curves $H = 0$ for $c > 1$

Once $y > 1$ and so $c(y) > 1$, we see from Figure 9 that it is possible to connect the state $q^- = \nabla u^- = e_2$ to another state q^+, namely the point on the negative x-axis, along which the two pieces of the hyperbolas meet. It is therefore possible to build an equivocal curve connecting these states, although note that *the characteristics do not leave this shock tangentially.* This is possible only since the Hamiltonian H is not C^1 and in particular the set Σ has a corner at q^+. See Figure 10, where we have marked for $c = 2$ velocity vectors for the characteristics entering and leaving the ES line. These geometric observations provide some partial insights into Isaacs' speculations [I, Section 10.8] as to when, and why, equivocal surfaces may be regarded as continuations of barriers.

A special case. Figure 10 shows the solution in the simple case that $c(y) \equiv 2$, and so $q^- = e_2$, $q^+ = -e_1$. The dynamics are

$$\begin{cases} \dot{x} = 2 + \cos\alpha \\ \dot{y} = 2\beta + \sin\alpha, \end{cases} \tag{6.14}$$

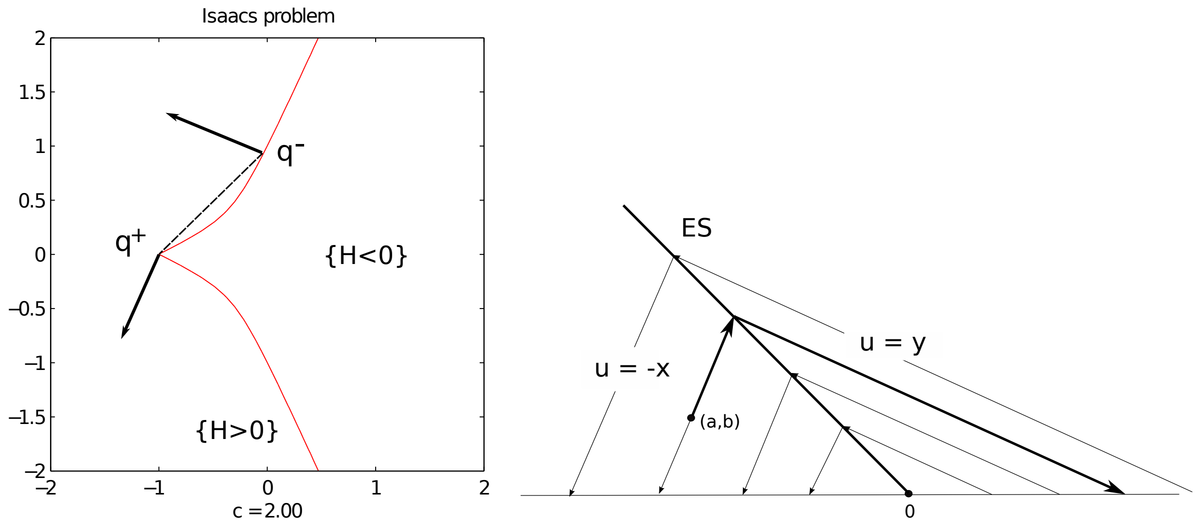

Figure 10: Solution for $c = 2$

the running payoff is $r \equiv 1$; and the HJI equation is

$$H(Du) = -2u_x - |Du| + 2|u_y| - 1 = 0 \quad \text{in } U \tag{6.15}$$

in the viscosity sense. There is no barrier and the solution is linear on either side of the line $ES = \{x+y = 0\}$. We have drawn as a heavy line the optimal path starting at a point (a, b) to the left of ES; it moves in the reverse direction of the characteristics ("retrograde" in Isaacs' terminology). It is easy to check that the optimal controls for the minimizing player are $\beta \equiv 1$ to the left of ES and $\beta \equiv -1$ to the right.

This is simple enough, but a paradox of sorts arises if we smooth the function H, by defining for small $\epsilon > 0$

$$H^\epsilon(p) := -2p_1 - |p| + 2(p_2^2 + \epsilon^2)^{\frac{1}{2}} - 1. \tag{6.16}$$

The HJI equation is then

$$H^\epsilon(Du^\epsilon) := -2u_x^\epsilon - |Du^\epsilon| + 2((u_y^\epsilon)^2 + \epsilon^2)^{\frac{1}{2}} - 1 = 0. \tag{6.17}$$

Since

$$(q^2 + \epsilon^2)^{\frac{1}{2}} = \sup_{|b| \le 1} \{qb + \epsilon(1 - b^2)^{\frac{1}{2}}\},$$

the PDE (6.17) corresponds to the differential game with dynamics (6.14) and running payoff

$$r^\epsilon := 1 - 2\epsilon(1 - b^2)^{\frac{1}{2}}. \tag{6.18}$$

The paradox is that *an equivocal curve does not exist for* $\epsilon > 0$. The geometric reason is that a left state q^- close to e_2 cannot be joined to a right state q^+ close to $-e_1$ for which the gradient $DH^\epsilon(q^+)$ points downward: see Figure 11. In fact the solution u^ϵ has a compressive shock, and the optimal trajectory starting from a point (a, b) is as drawn in Figure 11. This trajectory corresponds to the optimal strategy for the minimizing player that is $\beta \equiv -1$ until the trajectory hits the negative real axis, followed then by forcing by $\dot{y} = 0$. Fully understanding this phenomenon at the level of the HJI equation remains a challenge for nonlinear PDE theory.

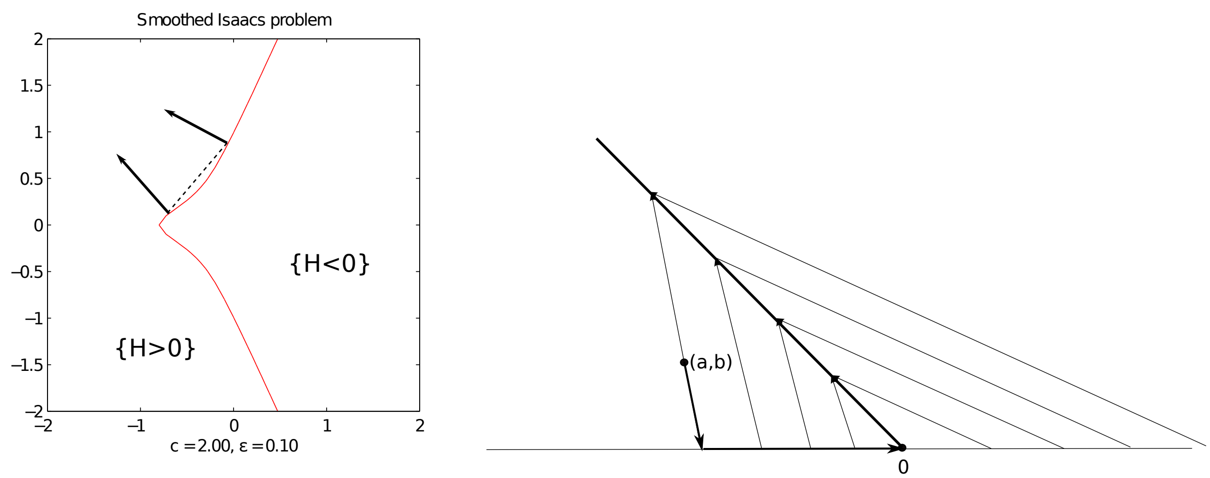

Figure 11: The smoothed Hamiltonian for $c = 2, \epsilon > 0$

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