Everywhere differentiability of infinity harmonic functions

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Abstract

We show that an infinity harmonic function, that is, a viscosity solution of the nonlinear PDE $-\Delta_\infty u = -u_{x_i}u_{x_j}u_{x_ix_j} = 0$, is everywhere differentiable. Our new innovation is proving the uniqueness of appropriately rescaled blow-up limits around an arbitrary point.

1 Introduction

We study in this paper differentiability properties of viscosity solutions of the PDE

$$-\Delta_\infty u = 0 \quad \text{in } U, \tag{1.1}$$

where $U \subseteq \mathbb{R}^n$ is an open set and we write

$$\Delta_\infty u := u_{x_i}u_{x_j}u_{x_ix_j}$$

for the *infinity-Laplacian* operator. This highly degenerate nonlinear PDE arises as a variational equation in the "calculus of variations in the sup-norm" (Crandall [C], Aronsson, Crandall and Juutinen [A-C-J]) and also appears in stochastic "tug-of-war" game theory (Peres, Schramm, Sheffield and Wilson [P-S-S-W]). A viscosity solution u is called an *infinity harmonic* function.

It is easy to show that a bounded viscosity solution is locally Lipschitz continuous and is consequently differentiable almost everywhere. We prove in this paper the regularity assertion that an infinity harmonic function is in fact everywhere differentiable.

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More precisely, let us assume that u is a viscosity solution of (1.1) and that the ball $B(x,r)$ lies in U. We then define

$$L_r^+(x) := \frac{\max_{\partial B(x,r)} u - u(x)}{r}, \;\; L_r^-(x) := \frac{u(x) - \min_{\partial B(x,r)} u}{r}.$$

As proved for example in [C-E-G], the limits

$$L(x) := \lim_{r\to 0} L_r^+(x) = \lim_{r\to 0} L_r^-(x)$$

exist and are equal for each point $x \in U$.

More interestingly, the paper [C-E-G] proves the following theorem, asserting that any blow-up limit around any point $x \in U$ must be a linear function.

THEOREM 1.1 *For each sequence $\{r_j\}_{j=1}^\infty$ converging to zero, there exists a subsequence $\{r_{j_k}\}_{k=1}^\infty$ such that*

$$\frac{u(r_{j_k} y + x) - u(x)}{r_{j_k}} \to a \cdot y \qquad \textit{locally uniformly,} \tag{1.2}$$

for some $a \in \mathbb{R}^n$ for which

$$|a| = L(x). \tag{1.3}$$

See [C-E] for a fairly simple proof.

Since solutions of $-\Delta_\infty u = 0$ are locally Lipschitz continuous, the rescaled functions $u_r(y) := \frac{u(ry+x)-u(x)}{r}$ are locally bounded and Lipschitz continuous and consequently contain a locally uniformly convergent subsequence. Theorem 1.1 asserts that each such limit is linear, but does not prove that various blow-up limits, possibly corresponding to different subsequences of radii going to zero, are the same (unless $L(x) = 0$). *Our main contribution in this paper is therefore establishing uniqueness for the blow-up limit*, from which it follows that the full limit

$$\lim_{r\to 0} \frac{u(ry+x) - u(x)}{r} = a \cdot y \tag{1.4}$$

exists locally uniformly, $a = Du(x)$ and $L(x) = |Du(x)|$.

The presentation in this paper is a simplification of our original proof, which will appear in our companion paper [E-S] on adjoint methods for the infinity Laplacian PDE. We have streamlined the current argument by eliminating various integral estimates involving Green's function for the linearization and replacing these with the one-sided derivative bound (2.12).

The price is the unmotivated introduction of the ad hoc expression (2.18) to which we apply the maximum principle.

O. Savin in [S] has shown that infinity harmonic functions in $n = 2$ variables are in fact continuously differentiable, and [E-Sv] proves the Hölder continuity of the gradient in two dimensions. It remains an open problem to determine if infinity harmonic functions are necessarily continuously differentiable for dimensions $n \geq 3$.

2 Estimates.

2.1 Gradient bounds. Assume for this section that U is bounded and that u is a bounded and Lipschitz continuous infinity harmonic function within U. We approximate by the smooth functions u^ε solving the regularized equations

$$\begin{cases} -\Delta_\infty u^\varepsilon - \varepsilon \Delta u^\varepsilon = 0 & \text{in } U \\ u^\varepsilon = u & \text{on } \partial U \end{cases} \tag{2.1}$$

for small $\varepsilon > 0$.

We will need a sup-norm estimate and a local gradient estimate, uniform in ε:

THEOREM 2.1 (i) *There exists a unique solution u^ε of* (2.1)*, smooth on $\bar{U}$. Furthermore, we have the estimates*

$$\max_{\bar{U}} |u^\varepsilon| \leq C, \tag{2.2}$$

and for each open set $V \subset\subset U$

$$\max_{\bar{V}} |Du^\varepsilon| \leq C. \tag{2.3}$$

(ii) *Furthermore,*

$$u^\varepsilon \to u \quad \text{uniformly on } \bar{U}. \tag{2.4}$$

Both constants C are independent of ε and the constant in (2.3) depends upon $\operatorname{dist}(V, \partial U)$.

Proof. 1. Existence of a smooth solution u^ε follows from standard quasilinear elliptic theory and the following a priori estimates. According to the maximum principle, $\max_{\bar{U}} |u^\varepsilon| = \max_{\partial U} |u^\varepsilon| = \max_{\partial U} |u|$.

2. Write

$$L_\varepsilon v := -u^\varepsilon_{x_i} u^\varepsilon_{x_j} v_{x_i x_j} - 2u^\varepsilon_{x_i} u^\varepsilon_{x_j x_j} v_{x_j} - \varepsilon \Delta v$$

for the linearization of PDE in (2.1). Since $L_\varepsilon u^\varepsilon_{x_k} = 0$ for $k = 1, \dots, n$, we calculate for $v^\varepsilon := \frac{1}{2}|Du^\varepsilon|^2$ that

$$L_\varepsilon v^\varepsilon = -u^\varepsilon_{x_i} u^\varepsilon_{x_j} v^\varepsilon_{x_i x_j} - 2u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} v^\varepsilon_{x_j} - \varepsilon \Delta v^\varepsilon = -(|D^2 u^\varepsilon Du^\varepsilon|^2 + \varepsilon |D^2 u^\varepsilon|^2). \tag{2.5}$$

Likewise for $z^\varepsilon := \frac{1}{2}(u^\varepsilon)^2$ we have the identity

$$\begin{aligned} L_\varepsilon z^\varepsilon &= -u^\varepsilon_{x_i} u^\varepsilon_{x_j} z^\varepsilon_{x_i x_j} - 2u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} z^\varepsilon_{x_j} - \varepsilon \Delta z^\varepsilon \\ &= -(|Du^\varepsilon|^4 + \varepsilon |Du^\varepsilon|^2) - 2u^\varepsilon u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} u^\varepsilon_{x_j}, \end{aligned} \tag{2.6}$$

according to (2.1). Select a smooth function ζ such that $\zeta \equiv 1$ on V, and $\zeta \equiv 0$ near ∂U, and put

$$w^\varepsilon := \zeta^2 v^\varepsilon + \alpha z^\varepsilon,$$

the constant $\alpha > 1$ to be selected.

Assume w^ε attains its maximum at an interior point $x^0 \in U$. In light of (2.5) and (2.6), at this point we have

$$\begin{aligned} 0 \le L_\varepsilon w^\varepsilon &= L_\varepsilon(\zeta^2 w^\varepsilon) + \alpha L_\varepsilon z^\varepsilon \\ &= \zeta^2 L_\varepsilon(w^\varepsilon) + \alpha L_\varepsilon z^\varepsilon + w^\varepsilon L_\varepsilon(\zeta^2) - 4u^\varepsilon_{x_i} u^\varepsilon_{x_j} w^\varepsilon_{x_j} \zeta \zeta_{x_i} - 4\varepsilon z^\varepsilon_{x_i} \zeta \zeta_{x_i} \\ &\le -\zeta^2(|D^2 u^\varepsilon Du^\varepsilon|^2 + \varepsilon |D^2 u^\varepsilon|^2) - \alpha(|Du^\varepsilon|^4 + \varepsilon |Du^\varepsilon|^2) \\ &\quad + C\varepsilon |D^2 u^\varepsilon Du^\varepsilon||Du^\varepsilon| + C(|Du^\varepsilon|^2 + \alpha)(|Du^\varepsilon|^2 + |D^2 u^\varepsilon Du^\varepsilon|\zeta) \\ &\quad + C|Du^\varepsilon|^2(|D^2 u^\varepsilon Du^\varepsilon| + \alpha |Du^\varepsilon|)\zeta + C\varepsilon |Du^\varepsilon| \\ &\le -\zeta^2(|D^2 u^\varepsilon Du^\varepsilon|^2 + \varepsilon |D^2 u^\varepsilon|^2) - \alpha(|Du^\varepsilon|^4 + \varepsilon |Du^\varepsilon|^2) \\ &\quad + C\varepsilon |D^2 u^\varepsilon Du^\varepsilon||Du^\varepsilon| + C|Du^\varepsilon|^4 + \tfrac{1}{2}\zeta^2 |D^2 u^\varepsilon Du^\varepsilon|^2 + C\alpha^3. \end{aligned}$$

If we now select α large enough, it follows that at x^0

$$\begin{aligned} \zeta^2 |D^2 u^\varepsilon Du^\varepsilon|^2 + |Du^\varepsilon|^4 &\le C + C\varepsilon |D^2 u^\varepsilon Du^\varepsilon||Du^\varepsilon| \\ &\le C + C|D^2 u^\varepsilon Du^\varepsilon|^{\frac{4}{3}} + \tfrac{1}{2}|Du^\varepsilon|^4. \end{aligned}$$

Therefore

$$\zeta^2 |D^2 u^\varepsilon Du^\varepsilon|^2 + |Du^\varepsilon|^4 \le C + C|D^2 u^\varepsilon Du^\varepsilon|^{\frac{4}{3}}.$$

Multiply by ζ^4 and as follows estimate:

$$\zeta^6 |D^2 u^\varepsilon Du^\varepsilon|^2 + \zeta^4 |Du^\varepsilon|^4 \le C + \zeta^4 C|D^2 u^\varepsilon Du^\varepsilon|^{\frac{4}{3}} \le C + \zeta^6 |D^2 u^\varepsilon Du^\varepsilon|^2.$$

We have thereby derived a bound on the term $\zeta^4 |Du^\varepsilon|^4$ at an interior point x^0 where $w^\varepsilon = \zeta^2 v^\varepsilon + \alpha z^\varepsilon = \frac{\zeta^2}{2}|Du^\varepsilon|^2 + \frac{\alpha}{2}(u^\varepsilon)^2$ attains its maximum. Since u^ε is bounded and $\zeta = 0$ on ∂U, we therefore have an L^∞-estimate on w^ε. The interior gradient bound (2.3) follows.

3. We must next study the behavior of u^ε near ∂U. To do so, select any point belonging to ∂U; without loss this point is 0. Fix a number $0 < \alpha < 1$ and define

$$w := \lambda |x|^\alpha.$$

The boundary function u is Lipschitz continuous, and consequently we can fix $\lambda > 0$ so large that

$$w + u(0) \geq u \quad \text{on } \partial U,$$

the constant λ depending only upon the local Lipschitz constant for u. Now compute

$$-\Delta_\infty w - \varepsilon \Delta w = \frac{\lambda^3 \alpha^3 (1-\alpha)}{|x|^{4-3\alpha}} - \frac{\lambda \varepsilon \alpha (n + \alpha - 2)}{|x|^{2-\alpha}}.$$

Since $4 - 3\alpha > 2 - \alpha$, we have

$$-\Delta_\infty w - \varepsilon \Delta w \geq 0$$

within U, provided ε is small enough. The maximum principle now lets us conclude that $w + u(0) \geq u^\varepsilon$ within U. Similarly, $-w + u(0) \leq u^\varepsilon$. Therefore

$$|u^\varepsilon(x) - u(0)| \leq \lambda |x|^\alpha \tag{2.7}$$

for all $x \in U$.

Using the analogous estimate at each boundary point and the interior gradient estimate (2.3), we deduce that a subsequence of $\{u^\varepsilon\}_{\varepsilon > 0}$ converges uniformly on $\bar{U}$ to a continuous limit function $\hat{u}$, which is infinity harmonic in U and which agrees with u on ∂U. Then $\hat{u} \equiv u$, by uniqueness: see Jensen [J], Armstrong and Smart [A-S]. This proves the assertion (2.4). □

2.2 Flatness estimates. For this subsection we assume that u is a Lipschitz continuous viscosity solution of the infinity Laplacian equation

$$-\Delta_\infty u = 0 \quad \text{in } B(0,3). \tag{2.8}$$

We as before introduce the regularizations

$$\begin{cases} -\Delta_\infty u^\varepsilon - \varepsilon \Delta u^\varepsilon = 0 & \text{in } B(0,3) \\ u^\varepsilon = u & \text{in } \partial B(0,3). \end{cases} \tag{2.9}$$

According to Theorem 2.1,

$$\max_{B(0,2)} |u^\varepsilon|, |Du^\varepsilon| \leq C \tag{2.10}$$

and $u^\varepsilon \to u$ uniformly.

We now make the additional "flatness" assumption that the functions u^ε are uniformly close to an affine function in $B(0,2)$, which we take to be the linear function x_n. So we henceforth suppose

$$\max_{B(0,2)} |u^\varepsilon - x_n| =: \lambda, \tag{2.11}$$

where λ is small.

THEOREM 2.2 *We have the pointwise, one-sided bound*

$$|Du^\varepsilon|^2 \leq u^\varepsilon_{x_n} + C(\lambda^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}/\lambda^{\frac{1}{2}}) \tag{2.12}$$

everywhere in $B(0,1)$.

The constant C does not depend upon ε.

Proof. 1. As before, write

$$L_\varepsilon v := -u^\varepsilon_{x_i} u^\varepsilon_{x_j} v_{x_i x_j} - 2u^\varepsilon_{x_i} u^\varepsilon_{x_j x_j} v_{x_j} - \varepsilon \Delta v$$

for the linearization of PDE in (2.9). We begin by determining how the operator L_ε acts upon various quadratic and quartic expressions involving u^ε and Du^ε.

As before, upon differentiating the PDE (2.9), we see that

$$L_\varepsilon u^\varepsilon_{x_k} = 0 \tag{2.13}$$

for $k = 1, \dots, n$. As in (2.5), this leads to the identity

$$L_\varepsilon(|Du^\varepsilon|^2) = -2|D^2 u^\varepsilon Du^\varepsilon|^2 - 2\varepsilon |D^2 u^\varepsilon|^2. \tag{2.14}$$

2. Now we compute

$$\begin{aligned} L_\varepsilon((u^\varepsilon - x_n)^2) = & -2(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2 - 2\varepsilon |Du^\varepsilon - e_n|^2 \\ & - 2(u^\varepsilon - x_n)(u^\varepsilon_{x_i} u^\varepsilon_{x_j} u^\varepsilon_{x_i x_j} + \varepsilon \Delta u^\varepsilon) \\ & - 2(u^\varepsilon - x_n) u^\varepsilon_{x_i} u^\varepsilon_{x_j x_j} (u^\varepsilon_{x_j} - \delta_{jn}). \end{aligned}$$

Here $e_n = (0, \dots, 1)$ is the unit vector in the x_n direction. Owing to the PDE (2.9), the flatness condition (2.11) and the estimates (2.10), we have

$$L_\varepsilon((u^\varepsilon - x_n)^2) \leq -2(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2 + C\lambda |D^2 u^\varepsilon Du^\varepsilon|. \tag{2.15}$$

3. Next put $\Phi(p) := (|p|^2 - p_n)_+^2$, where $x_+ := \max\{x, 0\}$. Then if $|p|^2 > p_n$, we have

$$\Phi_{p_k p_l} = 4(|p|^2 - p_n)\delta_{kl} + 2(2p_l - \delta_{ln})(2p_k - \delta_{kn}).$$

Therefore if at some point

$$|Du^\varepsilon|^2 - u^\varepsilon_{x_n} > 0, \tag{2.16}$$

we can multiply (2.13) by $\Phi_{p_k}(Du^\varepsilon)$ and sum on k, to discover after some computations that

$$\begin{aligned}
L_\varepsilon(\Phi(Du^\varepsilon)) &= -4(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})(|D^2u^\varepsilon Du^\varepsilon|^2 + \varepsilon|D^2u^\varepsilon|^2) \\
&\qquad - 2[(2\Delta_\infty u^\varepsilon - u^\varepsilon_{x_i}u^\varepsilon_{x_i x_n})^2 + \varepsilon(u^\varepsilon_{x_k}u^\varepsilon_{x_k x_i} - u^\varepsilon_{x_n x_i})(u^\varepsilon_{x_l}u^\varepsilon_{x_l x_i} - u^\varepsilon_{x_n x_i})] \\
&\le -4(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})|D^2u^\varepsilon Du^\varepsilon|^2 - 2(2\Delta_\infty u^\varepsilon - u^\varepsilon_{x_i}u^\varepsilon_{x_i x_n})^2.
\end{aligned}$$

Next select a smooth, nonnegative function ζ vanishing near $\partial B(0,2)$ such that $\zeta \equiv 1$ on $B(0,1)$. Then, again assuming the inequality (2.16), we have

$$\begin{aligned}
L_\varepsilon(\zeta^2\Phi(Du^\varepsilon)) &= \zeta^2 L_\varepsilon(\Phi(Du^\varepsilon)) + \Phi(Du^\varepsilon)L_\varepsilon(\zeta^2) - 4u^\varepsilon_{x_i}u^\varepsilon_{x_j}\Phi_{x_i}\zeta\zeta_{x_j} - 4\varepsilon\zeta\Phi_{x_i}\zeta_{x_i} \\
&\le -4\zeta^2(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})|D^2u^\varepsilon Du^\varepsilon|^2 - 2\zeta^2(2\Delta_\infty u^\varepsilon - u^\varepsilon_{x_i}u^\varepsilon_{x_i x_n})^2 \\
&\qquad + C(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2(1 + \zeta|D^2u^\varepsilon Du^\varepsilon|) \\
&\qquad + C\zeta(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})|2\Delta_\infty u^\varepsilon - u^\varepsilon_{x_i}u^\varepsilon_{x_i x_n}| + C\varepsilon|D^2u^\varepsilon| \\
&\le C(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2 + C\varepsilon|D^2u^\varepsilon|.
\end{aligned} \tag{2.17}$$

4. Now define

$$v^\varepsilon := \zeta^2\Phi(Du^\varepsilon) + \alpha(u^\varepsilon - x_n)^2 + \lambda|Du^\varepsilon|^2, \tag{2.18}$$

$\alpha > 0$ to be selected.

We assert that for a proper choice of the constant α, we have

$$v^\varepsilon \le C\left(\lambda + \frac{\varepsilon}{\lambda}\right) \quad \text{everywhere in } B(0,2). \tag{2.19}$$

To prove this, assume first that v^ε attains its maximum at an interior point x^0 of $B(0,2)$. If $|Du^\varepsilon|^2 - u^\varepsilon_{x_n} \le 0$ at x^0, then $\Phi(Du^\varepsilon) = 0$ there; and (2.19) follows from (2.10) and (2.11). Suppose instead that $|Du^\varepsilon|^2 - u^\varepsilon_{x_n} > 0$ at x^0. We then employ (2.14), (2.15) and (2.17) to calculate at the point x^0 that

$$\begin{aligned}
0 \le L_\varepsilon(v^\varepsilon) &= L_\varepsilon(\zeta^2\Phi(Du^\varepsilon)) + \alpha L_\varepsilon((u^\varepsilon - x_n)^2) + \lambda L_\varepsilon(|Du^\varepsilon|^2) \\
&\le C(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2 + C\varepsilon|D^2u^\varepsilon| \\
&\qquad + \alpha(-2(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2 + C\lambda|D^2u^\varepsilon Du^\varepsilon|) \\
&\qquad + \lambda(-2|D^2u^\varepsilon Du^\varepsilon|^2 - 2\varepsilon|D^2u^\varepsilon|^2).
\end{aligned}$$

We adjust α large enough, to deduce that the inequality

$$(|Du^\varepsilon|^2 - u^\varepsilon_{x_n})^2_+ \leq C(\lambda + \varepsilon/\lambda)$$

holds at the maximum point x^0. This implies (2.19).

Suppose lastly that v^ε attains its maximum only on the boundary $\partial B(0,2)$. Since $\zeta = 0$ there, we again deduce (2.19).

Since $\zeta \equiv 1$ on $B(0,1)$, (2.19) implies the interior estimate (2.12). □

3 Everywhere differentiability

This section employs the one-sided bound (2.12) to prove the uniqueness of blow-up limits.

For later use we first record a simple observation:

LEMMA 3.1 *Assume $b \in \mathbb{R}^n$, $|b| = 1$. Let v be a smooth function satisfying*

$$\max_{B(0,1)} |v - b \cdot x| \leq \eta$$

for some constant η. Then there exists a point $x^0 \in B(0,1)$ at which

$$|Dv(x^0) - b| \leq 4\eta$$

Proof. Define

$$w := b \cdot x - 2\eta|x|^2.$$

Then $(v - w)(0) \leq \eta$. Furthermore, if $x \in \partial B(0,1)$, then $(v - w)(x) = v - b \cdot x + 2\eta \geq \eta$. Consequently $v - w$ attains its minimum over $B(0,1)$ at some interior point x^0, at which $Dv(x^0) = Dw(x^0) = b - 4\eta x^0$. □

Our main result is this:

THEOREM 3.2 *Let u be a viscosity solution of*

$$-\Delta_\infty u = 0 \quad \text{in } U. \tag{3.1}$$

Then u is differentiable at each point in U.

Proof. 1. Select any point within U, which without loss we may assume is 0. Suppose that the blow up discussed in §1 does not produce a unique tangent plane at 0. This means there exist two sequences of radii $\{r_j\}_{j=1}^{\infty}, \{s_k\}_{k=1}^{\infty}$, each converging to zero, for which

$$\max_{B(0,r_j)} \frac{1}{r_j}|u(x) - u(0) - a \cdot x| \to 0 \tag{3.2}$$

and

$$\max_{B(0,s_k)} \frac{1}{s_k}|u(x) - u(0) - b \cdot x| \to 0 \tag{3.3}$$

for distinct vectors $a, b \in \mathbb{R}^n$, with $|a| = |b| > 0$. We may assume without loss that

$$a = e_n,\ |b| = 1,\ b \neq e_n.$$

Write $b = (b_1, \dots, b_n)$ and define

$$\theta := 1 - b_n > 0. \tag{3.4}$$

2. Hereafter C denotes the constant from (2.12). Select $\lambda > 0$ so small that

$$2C\lambda^{\frac{1}{2}} = \frac{\theta}{4}. \tag{3.5}$$

Put

$$\varepsilon_1 = \lambda^2. \tag{3.6}$$

3. We next use (3.2) (with $a = e_n$) to select a radius $r > 0$ for which

$$\max_{B(0,r)} \frac{1}{r}|u(x) - u(0) - x_n| \leq \frac{\lambda}{4}.$$

We may without loss assume that $r = 2$ and that $u(0) = 0$, as we can otherwise rescale and consider the function $\frac{u(rx)-u(0)}{r}$. Hence

$$\max_{B(0,2)} |u - x_n| \leq \frac{\lambda}{2}. \tag{3.7}$$

Now fix $\varepsilon_2 > 0$ so small that

$$\max_{B(0,2)} |u^{\varepsilon} - x_n| \leq \lambda \tag{3.8}$$

for all $0 < \varepsilon \leq \varepsilon_2$.

We introduce yet another constant $\eta > 0$, picked so that

$$12\eta = \frac{\theta}{4}. \tag{3.9}$$

In view of (3.3), we can find a (possibly very small) radius $0 < s < 1$ for which

$$\max_{B(0,s)} \frac{1}{s}|u - b \cdot x| \leq \frac{\eta}{2}.$$

We select $\varepsilon_3 > 0$ so that

$$\max_{B(0,s)} \frac{1}{s}|u^\varepsilon - b \cdot x| \leq \eta \tag{3.10}$$

for all $0 < \varepsilon \leq \varepsilon_3$.

Hereafter let

$$\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}. \tag{3.11}$$

4. Rescaling (3.10) to the unit ball and applying the Lemma, we secure a point $x^0 \in B(0,s) \subseteq B(0,1)$ at which

$$|Du^\varepsilon(x^0) - b| \leq 4\eta.$$

Then

$$|u^\varepsilon_{x_n}(x^0) - b_n| \leq 4\eta; \tag{3.12}$$

and since $|b| = 1$, we also have

$$|Du^\varepsilon(x^0)| \geq 1 - 4\eta. \tag{3.13}$$

We now use (2.12), the choice (3.5) of λ and the choice (3.6) of ε_1, to deduce

$$|Du^\varepsilon(x^0)|^2 \leq u^\varepsilon_{x_n}(x^0) + C(\lambda^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}/\lambda^{\frac{1}{2}}) \leq u^\varepsilon_{x_n}(x^0) + \frac{\theta}{4}.$$

But (3.12) and (3.13) imply

$$(1 - 4\eta)^2 \leq b_n + 4\eta + \frac{\theta}{4};$$

whence

$$\theta = 1 - b_n \leq 12\eta + \frac{\theta}{4} = \frac{\theta}{2},$$

in view of (3.9). This is a contradiction since $\theta > 0$. □

4 An integral estimate for the gradient

We next discover an integral estimate on the deviation of Du from the slope of an approximating linear function, and from this deduce that every point is a Lebesgue point for the gradient.

THEOREM 4.1 *Suppose that u is a viscosity solution of*

$$-\Delta_\infty u = 0 \qquad \text{in } B(0,2)$$

and that

$$\max_{B(0,2)} |u - u(0) - a \cdot x| =: \lambda \tag{4.1}$$

is small. Then we have the integral estimate

$$\int_{B(0,1)} |Du - a|^2 \, dx \leq C\lambda, \tag{4.2}$$

in which the constant C depends upon $|a|$.

We can regard (4.2) as a crude sort of "Caccioppoli inequality" for solutions of the infinity Laplacian PDE: see Giaquinta [G].

Proof. Suppose $u(0) = 0$. From standard comparison with cones arguments (see [C-E-G]) we see that

$$\max_{B(0,1)} |Du| \leq |a| + C\lambda. \tag{4.3}$$

We may assume $a = |a| e_n$. Let L denote a line segment within $B(0,1)$ in the e_n direction, with endpoints $y^{\pm} \in \partial B(0,1)$. Then (4.3) implies

$$\begin{aligned}
\int_L |u_{x_n} - |a|| \, dx_n &\leq \int_L |u_{x_n} - (|a| + C\lambda)| \, dx_n + C\lambda \\
&= \int_L |a| + C\lambda - u_{x_n} \, dx_n + C\lambda \\
&\leq (|a| y_n^+ - u(y^+)) - (|a| y_n^- - u(y^-)) + C\lambda \\
&\leq C\lambda.
\end{aligned}$$

We again use (4.3) to observe also that

$$
\begin{aligned}
\int_L |D'u|^2 \, dx_n &\leq \int_L |a|^2 - (u_{x_n})^2 \, dx_n + C\lambda \\
&\leq C \int_L |u_{x_n} - |a|| \, dx_n + C\lambda \\
&\leq C\lambda,
\end{aligned}
$$

for $D'u := (u_{x_1}, \ldots, u_{x_{n-1}}, 0)$. Integrating now over all such vertical line segments L within the ball $B(0,1)$, we deduce (4.2). □

As an application of the foregoing estimate, we have:

THEOREM 4.2 *Let u be a viscosity solution of*

$$
-\Delta_\infty u = 0 \quad \textit{in } U.
$$

Then each point $x^0 \in U$ is a Lebesgue point for Du.

Proof. We may assume that a given point x^0 in U is the origin. Select any small number $\lambda > 0$. It follows from Theorem 3.2 that upon rescaling we may assume that the flatness condition (4.1) is valid for $a = Du(0)$.

The previous theorem provides us with the inequality (4.2). Given any preassigned small number $\gamma > 0$, we select λ so small that the term on the left of (4.2) is no greater than γ. □

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