

New Identities for Weak KAM Theory

Lawrence C. Evans*
Department of Mathematics
University of California, Berkeley

Abstract

This paper records for the Hamiltonian $H = \frac{1}{2}|p|^2 + W(x)$ some old and new identities relevant for the PDE/variational approach to weak KAM theory.

For Haim Brezis, in continuing admiration.

1 Introduction

1.1 Weak KAM for a model Hamiltonian. This is a follow-up to two of my earlier papers [E1], [E2] that propose a PDE/variational approach to weak KAM theory, originating with Mather and Fathi ([M1], [M2] [F1], [F2], etc.) In this paper we specialize to the classical Hamiltonian

$$(1.1) \quad H(p, x) = \frac{1}{2}|p|^2 + W(x),$$

where the potential W is smooth and \mathbb{T}^n -periodic, where $\mathbb{T}^n = [0, 1]^n$ denotes the unit cube with opposite faces identified. Given a vector $P = (P_1, \dots, P_n) \in \mathbb{R}^n$, the corresponding *cell* PDE reads

$$(1.2) \quad \frac{1}{2}|P + Dv|^2 + W = \bar{H}(P) \quad \text{in } \mathbb{T}^n,$$

where $\bar{H} : \mathbb{R}^n \rightarrow \mathbb{R}$ is the *effective Hamiltonian* corresponding to H , as introduced in the important, but unpublished, paper of Lions–Papanicolaou–Varadhan [L-P-V]. Here $v = v(P, x)$ denotes a \mathbb{T}^n -periodic viscosity solution. As shown for instance in [E1] there exists also a Radon probability measure σ on \mathbb{T}^n solving the *transport PDE*

$$(1.3) \quad \operatorname{div}(\sigma(P + Dv)) = 0 \quad \text{in } \mathbb{T}^n$$

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in an appropriate weak sense.

A central goal of weak KAM theory is developing a nonperturbative methods to identify “integrable structures” within the otherwise possibly chaotic dynamics generated by a given Hamiltonian $H = H(p, x)$, and in particular to understand if and how the effective Hamiltonian \bar{H} encodes such information. The PDE approach to weak KAM aims at extracting such information from the two coupled PDE (1.2), (1.3).

This paper extends previous work by discovering for the particular case of the Hamiltonian (1.1) several new integral identities, especially for the variational approximations introduced below. We also record how some previously derived general formulas simplify in this case, and provide in Section 4 some applications.

I thank the referees for careful reading and in particular for finding an error in one of my proofs, now removed.

1.2 Variational approximation. We consider for fixed $\varepsilon > 0$ the problem of minimizing the functional

$$I_\varepsilon[v] := \int_{\mathbb{T}^n} e^{\frac{H(P+Dv, x)}{\varepsilon}} dx,$$

among Lipschitz continuous functions $v : \mathbb{T}^n \rightarrow \mathbb{R}$ with mean zero: $\int_{\mathbb{T}^n} v dx = 0$. We write $D = D_x$ to denote the gradient in the variables x . PDE and calculus of variations theory (see [E1]) implies that this problem has a unique, smooth minimizer $v^\varepsilon = v^\varepsilon(P, x)$, which satisfies therefore the Euler-Lagrange PDE

$$(1.4) \quad \operatorname{div}\left(e^{\frac{H(P+Dv^\varepsilon, x)}{\varepsilon}}(P + Dv^\varepsilon)\right) = 0 \quad \text{in } \mathbb{T}^n.$$

Standard regularity theory shows that v^ε is a smooth function of x and also of the parameters ε and $P = (P_1, \dots, P_n)$.

It is convenient to change notation, writing

$$(1.5) \quad \begin{cases} H^\varepsilon := H(P + Dv^\varepsilon, x) = \frac{1}{2}|P + Dv^\varepsilon|^2 + W \\ \bar{H}^\varepsilon(P) := \varepsilon \log \left(\int_{\mathbb{T}^n} e^{\frac{H^\varepsilon}{\varepsilon}} dx \right) \\ \sigma^\varepsilon := e^{\frac{H^\varepsilon - \bar{H}^\varepsilon(P)}{\varepsilon}}. \end{cases}$$

THEOREM 1.1 (i) *We have $\sigma^\varepsilon \geq 0$,*

$$(1.6) \quad \int_{\mathbb{T}^n} \sigma^\varepsilon dx = 1,$$

$$(1.7) \quad \operatorname{div}(\sigma^\varepsilon(P + Dv^\varepsilon)) = 0,$$

and

$$(1.8) \quad (P + Dv^\varepsilon) \cdot DH^\varepsilon + \varepsilon \Delta v^\varepsilon = (P_i + v_{x_i}^\varepsilon)(P_j + v_{x_j}^\varepsilon)v_{x_i x_j}^\varepsilon + (P_i + v_{x_i}^\varepsilon)W_{x_i} + \varepsilon \Delta v^\varepsilon = 0.$$

(ii) Furthermore,

$$(1.9) \quad \bar{H}^\varepsilon(P) \leq \bar{H}(P) \quad (P \in \mathbb{R}^n)$$

for each $\varepsilon > 0$, and

$$(1.10) \quad \lim_{\varepsilon \rightarrow 0} \bar{H}^\varepsilon(P) = \bar{H}(P) \quad (P \in \mathbb{R}^n).$$

As above, $D = D_x$ means the first derivatives in x , and $D^2 = D_x^2$ the second derivatives in x . Likewise, $\Delta = \Delta_x$ is the Laplacian in the x -variables.

Proof. The term $\bar{H}^\varepsilon(P)$ is introduced to achieve the normalization (1.6). The PDE (1.7) and (1.8) are the Euler-Lagrange equation (1.4) rewritten in respective divergence and non-divergence forms.

The assertion (1.9) follows upon our using a solution of (1.2) in the variational principle, and the limit (1.10) is demonstrated in [E1]. \square

Remark. As shown in [E1], we have the uniform estimates

$$\max_{\mathbb{T}^n} |D_x v^\varepsilon|, |v^\varepsilon| \leq C$$

for a constant C independent of ε . Hence we can extract a subsequence such that

$$v^{\varepsilon_k} \rightarrow v \text{ uniformly on } \mathbb{T}^n, \quad \sigma^{\varepsilon_k} \rightharpoonup \sigma \text{ weakly as measures.}$$

A main assertion of [E1] is that v, σ solve the transport equation (1.3) and the cell PDE (1.2) on the support of σ . In particular Dv makes sense σ -almost everywhere, even if σ has a singular part with respect to Lebesgue measure. \square

See also Bernardi–Cardin–Guzzo [B-C-G], Gomes–Sanchez Morgado [G-SM], Gomes – Iturriaga–Sanchez Morgado–Yu [G-I-SM-Y], etc for more on this variational method.

2 Identities and estimates

The ideas is to extract useful information from the two forms (1.7), (1.8) of the Euler–Lagrange PDE. This section records various relevant integral identities, mostly derived by differentiating with respect to different variables. Some of the resulting formulas are special cases of those in [E1], [E2] and some are new.

2.1 Differentiations in x . We start by differentiating with respect to x_k for $k = 1, \dots, n$:

THEOREM 2.1 *We have the identities*

$$(2.1) \quad \int_{\mathbb{T}^n} DW \sigma^\varepsilon dx = 0, \quad \int_{\mathbb{T}^n} \Delta v^\varepsilon \sigma^\varepsilon dx = 0$$

and

$$(2.2) \quad \int_{\mathbb{T}^n} (|D^2 v^\varepsilon|^2 + \Delta W) \sigma^\varepsilon + \varepsilon \frac{|D\sigma^\varepsilon|^2}{\sigma^\varepsilon} dx = 0.$$

Proof. In view of (1.1) and (1.5), we have

$$(2.3) \quad \frac{1}{2}|P + Dv^\varepsilon|^2 + W = \bar{H}^\varepsilon(P) + \varepsilon \log \sigma^\varepsilon.$$

Differentiating in x_k once, and then twice, we learn that

$$(2.4) \quad (P_i + v_{x_i}^\varepsilon) v_{x_i x_k}^\varepsilon + W_{x_k} = \varepsilon \frac{\sigma_{x_k}^\varepsilon}{\sigma^\varepsilon}$$

and

$$(2.5) \quad (P + Dv^\varepsilon) \cdot D(\Delta v^\varepsilon) + |D^2 v^\varepsilon|^2 + \Delta W = \varepsilon \operatorname{div}_x \left(\frac{D\sigma^\varepsilon}{\sigma^\varepsilon} \right).$$

Multiply (2.4) by σ^ε , integrate by parts and recall (1.7) to derive the first identity in (2.1). The second follows upon our multiplying (1.8) by σ^ε and integrating. To get (2.2), multiply (2.5) by σ^ε and integrate. \square

Remark. As $D\sigma^\varepsilon = \frac{1}{\varepsilon} DH^\varepsilon \sigma^\varepsilon$, we obtain from (2.2) the estimate

$$(2.6) \quad \int_{\mathbb{T}^n} \left(|D^2 v^\varepsilon|^2 + \frac{1}{\varepsilon} |DH^\varepsilon|^2 \right) \sigma^\varepsilon dx \leq C,$$

for a constant C independent of ε . Recall that we write $DH^\varepsilon = D_x H^\varepsilon$. \square

We next generalize Theorem 2.1:

THEOREM 2.2 *For each smooth function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, we have the identity*

$$(2.7) \quad \int_{\mathbb{T}^n} |D^2 v^\varepsilon|^2 \phi(\sigma^\varepsilon) \sigma^\varepsilon + (\Delta v^\varepsilon)^2 \phi'(\sigma^\varepsilon) (\sigma^\varepsilon)^2 dx \\ + \frac{1}{\varepsilon} \int_{\mathbb{T}^n} |DH^\varepsilon|^2 [\phi(\sigma^\varepsilon) \sigma^\varepsilon + \phi'(\sigma^\varepsilon) (\sigma^\varepsilon)^2] dx = - \int_{\mathbb{T}^n} \Delta W \phi(\sigma^\varepsilon) \sigma^\varepsilon dx.$$

Proof. 1. We multiply the Euler–Lagrange equation (1.7) by $\operatorname{div}(\phi(\sigma^\varepsilon)(P + Dv^\varepsilon))$ and integrate by parts over \mathbb{T}^n :

$$\begin{aligned}
0 &= \int_{\mathbb{T}^n} (\sigma^\varepsilon(P + v_{x_i}^\varepsilon))_{x_i} (\phi(\sigma^\varepsilon)(P + v_{x_j}^\varepsilon))_{x_j} dx \\
&= \int_{\mathbb{T}^n} (\sigma^\varepsilon(P + v_{x_i}^\varepsilon))_{x_j} (\phi(\sigma^\varepsilon)(P + v_{x_j}^\varepsilon))_{x_i} dx \\
&= \int_{\mathbb{T}^n} (\sigma_{x_j}^\varepsilon(P + v_{x_i}^\varepsilon) + \sigma^\varepsilon v_{x_i x_j}^\varepsilon) (\phi'(\sigma^\varepsilon) \sigma_{x_i}^\varepsilon(P + v_{x_j}^\varepsilon) + \phi(\sigma^\varepsilon) v_{x_i x_j}^\varepsilon) dx \\
&= \int_{\mathbb{T}^n} |D^2 v^\varepsilon|^2 \phi(\sigma^\varepsilon) \sigma^\varepsilon + \phi'(\sigma^\varepsilon) |D\sigma^\varepsilon \cdot (P + Dv^\varepsilon)|^2 \\
&\quad + [\phi(\sigma^\varepsilon) + \sigma^\varepsilon \phi'(\sigma^\varepsilon)] v_{x_i x_j}^\varepsilon (P + v_{x_j}^\varepsilon) \sigma_{x_i}^\varepsilon dx.
\end{aligned}$$

Now $D\sigma^\varepsilon \cdot (P + Dv^\varepsilon) = -\sigma^\varepsilon \Delta v^\varepsilon$, according to (1.7), and furthermore $v_{x_j x_i}^\varepsilon (P + v_{x_j}^\varepsilon) = H_{x_i}^\varepsilon - W_{x_i}$. We can therefore simplify, obtaining the identity

$$0 = \int_{\mathbb{T}^n} |D^2 v^\varepsilon|^2 \phi(\sigma^\varepsilon) \sigma^\varepsilon + \phi'(\sigma^\varepsilon) (\sigma^\varepsilon)^2 (\Delta v^\varepsilon)^2 + [\phi(\sigma^\varepsilon) + \sigma^\varepsilon \phi'(\sigma^\varepsilon)] (DH^\varepsilon - DW) \cdot D\sigma^\varepsilon dx.$$

Since $D\sigma^\varepsilon = \frac{1}{\varepsilon} DH^\varepsilon \sigma^\varepsilon$, it follows that

$$\begin{aligned}
&\int_{\mathbb{T}^n} |D^2 v^\varepsilon|^2 \phi(\sigma^\varepsilon) \sigma^\varepsilon + (\Delta v^\varepsilon)^2 \phi'(\sigma^\varepsilon) (\sigma^\varepsilon)^2 dx + \frac{1}{\varepsilon} \int_{\mathbb{T}^n} |DH^\varepsilon|^2 [\phi(\sigma^\varepsilon) \sigma^\varepsilon + \phi'(\sigma^\varepsilon) (\sigma^\varepsilon)^2] dx \\
&= \int_{\mathbb{T}^n} [\phi(\sigma^\varepsilon) + \sigma^\varepsilon \phi'(\sigma^\varepsilon)] DW \cdot D\sigma^\varepsilon dx = - \int_{\mathbb{T}^n} \Delta W \sigma^\varepsilon \phi(\sigma^\varepsilon) dx.
\end{aligned}$$

□

2.2 Differentiations in P. We next differentiate with respect to the parameters P_k , for $k = 1, \dots, n$. In the following expressions we write

$$(2.8) \quad V^\varepsilon := D\bar{H}^\varepsilon(P).$$

Hereafter D_P means the gradient in P , and $D_{x,P}^2$ means the mixed second derivatives in x and P . To minimize notational clutter, we can safely write $D\bar{H}^\varepsilon = D_P \bar{H}^\varepsilon$ and $D^2 \bar{H}^\varepsilon = D_P^2 \bar{H}^\varepsilon$, since $\bar{H}^\varepsilon = \bar{H}^\varepsilon(P)$ does not depend upon x .

THEOREM 2.3 *These further identities hold:*

$$(2.9) \quad D\bar{H}^\varepsilon(P) = V^\varepsilon = \int_{\mathbb{T}^n} (P + Dv^\varepsilon) \sigma^\varepsilon dx,$$

$$(2.10) \quad D^2 \bar{H}^\varepsilon(P) = \int_{\mathbb{T}^n} (I + D_{x,P}^2 v^\varepsilon) \otimes (I + D_{x,P}^2 v^\varepsilon) \sigma^\varepsilon + \varepsilon \frac{D_P \sigma^\varepsilon \otimes D_P \sigma^\varepsilon}{\sigma^\varepsilon} dx.$$

Remark. Formula (2.10) means that for $k, l = 1, \dots, n$,

$$\bar{H}_{P_k P_l}^\varepsilon(P) = \int_{\mathbb{T}^n} (\delta_{ik} + v_{x_i P_k}^\varepsilon)(\delta_{il} + v_{x_i P_l}^\varepsilon) \sigma^\varepsilon + \varepsilon \frac{\sigma_{P_k}^\varepsilon \sigma_{P_l}^\varepsilon}{\sigma^\varepsilon} dx.$$

Therefore $\xi \cdot D^2 \bar{H}(P) \xi = \bar{H}_{P_k P_l}^\varepsilon(P) \xi_k \xi_l \geq 0$ for all $\xi = (\xi_1, \dots, \xi_n)$ and hence

$$(2.11) \quad P \mapsto \bar{H}^\varepsilon(P) \quad \text{is convex.}$$

□

Proof. 1. Differentiating (2.3) in P_k , and then in P_l , we find

$$(2.12) \quad (P_i + v_{x_i}^\varepsilon)(\delta_{ik} + v_{x_i P_k}^\varepsilon) = \bar{H}_{P_k}^\varepsilon + \varepsilon \frac{\sigma_{P_k}^\varepsilon}{\sigma^\varepsilon},$$

$$(2.13) \quad (P_i + v_{x_i}^\varepsilon) v_{x_i P_k P_l}^\varepsilon + (\delta_{il} + v_{x_i P_l}^\varepsilon)(\delta_{ik} + v_{x_i P_k}^\varepsilon) = \bar{H}_{P_k P_l}^\varepsilon + \varepsilon \left(\frac{\sigma_{P_k}^\varepsilon}{\sigma^\varepsilon} \right)_{P_l}.$$

2. Since $\int_{\mathbb{T}^n} \sigma^\varepsilon dx = 1$, we have

$$(2.14) \quad 0 = \int_{\mathbb{T}^n} \sigma_{P_k}^\varepsilon dx = \int_{\mathbb{T}^n} \frac{\sigma_{P_k}^\varepsilon}{\sigma^\varepsilon} \sigma^\varepsilon dx.$$

We now multiply (2.12) by σ^ε and integrate, using (1.6), (1.7) and (2.14) to derive (2.9).

In addition, (2.14) implies

$$\int_{\mathbb{T}^n} \left(\frac{\sigma_{P_k}^\varepsilon}{\sigma^\varepsilon} \right)_{P_l} \sigma^\varepsilon dx = - \int_{\mathbb{T}^n} \frac{\sigma_{P_k}^\varepsilon \sigma_{P_l}^\varepsilon}{\sigma^\varepsilon} dx.$$

So the identity (2.10) follows, if we multiply (2.13) by σ^ε and integrate. □

Remark. It follows from (2.10) that

$$(2.15) \quad \begin{aligned} \text{tr}(D^2 \bar{H}^\varepsilon(P)) &= \int_{\mathbb{T}^n} |I + D_{x,P}^2 v^\varepsilon|^2 \sigma^\varepsilon + \varepsilon \frac{|D_P \sigma^\varepsilon|^2}{\sigma^\varepsilon} dx \\ &= \int_{\mathbb{T}^n} \left(|I + D_{x,P}^2 v^\varepsilon|^2 + \frac{1}{\varepsilon} |(P + Dv^\varepsilon)(I + D_{x,P}^2 v^\varepsilon) - V^\varepsilon|^2 \right) \sigma^\varepsilon dx \end{aligned}$$

□

where “tr” means trace.

2.3 Differentiations in ε . In the following, subscripts ε denote derivatives with respect to ε :

THEOREM 2.4 *We have*

$$(2.16) \quad \bar{H}_\varepsilon^\varepsilon(P) = - \int_{\mathbb{T}^n} \sigma^\varepsilon \log \sigma^\varepsilon dx = - \int_{\mathbb{T}^n} \frac{H^\varepsilon - \bar{H}^\varepsilon(P)}{\varepsilon} \sigma^\varepsilon dx,$$

and so

$$(2.17) \quad \int_{\mathbb{T}^n} H^\varepsilon \sigma^\varepsilon dx = \bar{H}^\varepsilon(P) - \varepsilon \bar{H}_\varepsilon^\varepsilon(P).$$

In addition,

$$(2.18) \quad \begin{aligned} \bar{H}_{\varepsilon\varepsilon}^\varepsilon(P) &= \int_{\mathbb{T}^n} |Dv_\varepsilon^\varepsilon|^2 \sigma^\varepsilon + \varepsilon \frac{|\sigma_\varepsilon^\varepsilon|^2}{\sigma^\varepsilon} dx \\ &= \int_{\mathbb{T}^n} \left(|Dv_\varepsilon^\varepsilon|^2 + \frac{1}{\varepsilon} \left| (P + Dv^\varepsilon) \cdot Dv_\varepsilon^\varepsilon - \frac{H^\varepsilon - \bar{H}^\varepsilon(P) + \varepsilon \bar{H}_\varepsilon^\varepsilon(P)}{\varepsilon} \right|^2 \right) \sigma^\varepsilon dx. \end{aligned}$$

Remark. The identity (2.18) implies

$$(2.19) \quad \varepsilon \mapsto \bar{H}^\varepsilon(P) \quad \text{is convex.}$$

Differentiating in x and then in P , we can likewise show that

$$(2.20) \quad (P, \varepsilon) \mapsto \bar{H}^\varepsilon(P) \quad \text{is jointly convex.}$$

□

Proof. 1. We differentiate (2.3) twice in ε , to learn that

$$(2.21) \quad (P + Dv^\varepsilon) \cdot Dv_\varepsilon^\varepsilon = \bar{H}_\varepsilon^\varepsilon(P) + \log \sigma^\varepsilon + \varepsilon \frac{\sigma_\varepsilon^\varepsilon}{\sigma^\varepsilon}$$

and then

$$(2.22) \quad (P + Dv^\varepsilon) \cdot Dv_{\varepsilon\varepsilon}^\varepsilon + |Dv_\varepsilon^\varepsilon|^2 = \bar{H}_{\varepsilon\varepsilon}^\varepsilon(P) + \frac{\sigma_\varepsilon^\varepsilon}{\sigma^\varepsilon} + \varepsilon \left(\frac{\sigma_\varepsilon^\varepsilon}{\sigma^\varepsilon} \right)_\varepsilon.$$

Multiply (2.21) by σ^ε and recall (1.7), to derive (2.16).

2. Next multiply (2.22) by σ^ε . We observe that

$$0 = \int_{\mathbb{T}^n} \sigma_\varepsilon^\varepsilon dx = \int_{\mathbb{T}^n} \frac{\sigma_\varepsilon^\varepsilon}{\sigma^\varepsilon} \sigma^\varepsilon dx;$$

and thus

$$\int_{\mathbb{T}^n} \left(\frac{\sigma_\varepsilon^\varepsilon}{\sigma^\varepsilon} \right)_\varepsilon \sigma^\varepsilon dx = - \int_{\mathbb{T}^n} \frac{|\sigma_\varepsilon^\varepsilon|^2}{\sigma^\varepsilon} dx.$$

This gives the first equality in (2.18), and the second follows when we explicitly calculate $\sigma_\varepsilon^\varepsilon$. \square

2.5 Estimates for $Du^\varepsilon - Du$. A key question is how well v^ε and σ^ε approximate as $\varepsilon \rightarrow 0$ particular solutions v, σ of the weak KAM PDE (1.2), (1.3).

Now let v be a viscosity solution of (1.2) and σ a corresponding weak solution of (1.3). To allow for changes in P we also assume, for this subsection only, that v^ε solves the variational problem for the vector P^ε . Consequently we have

$$\frac{1}{2}|P^\varepsilon + Dv^\varepsilon|^2 + W = \bar{H}^\varepsilon(P^\varepsilon) + \varepsilon \log \sigma^\varepsilon.$$

THEOREM 2.5 *These identities hold:*

$$(2.23) \quad \frac{1}{2} \int_{\mathbb{T}^n} |(P^\varepsilon + Dv^\varepsilon) - (P + Dv)|^2 \sigma^\varepsilon dx = \bar{H}(P) - \bar{H}^\varepsilon(P^\varepsilon) + \varepsilon \bar{H}_\varepsilon^\varepsilon(P^\varepsilon) + D\bar{H}^\varepsilon(P^\varepsilon) \cdot (P^\varepsilon - P)$$

and

$$(2.24) \quad \frac{1}{2} \int_{\mathbb{T}^n} |(P^\varepsilon + Dv^\varepsilon) - (P + Dv)|^2 d\sigma = \int_{\mathbb{T}^n} H^\varepsilon d\sigma - \bar{H}(P) + D\bar{H}(P) \cdot (P - P^\varepsilon).$$

Observe that right hand sides involve Taylor expansions of \bar{H}^ε and \bar{H} . Thus if \bar{H}^ε approximates \bar{H} sufficiently well for small ε and if P^ε is close to P , then $|Dv^\varepsilon - Dv|$ is small on the support of σ^ε .

Proof. 1. We have

$$\frac{1}{2}|P^\varepsilon + Dv^\varepsilon|^2 + W = \bar{H}^\varepsilon(P^\varepsilon) + \varepsilon \log \sigma^\varepsilon, \quad \frac{1}{2}|P + Dv|^2 + W = \bar{H}(P);$$

and so

$$(2.25) \quad \frac{1}{2}(|P^\varepsilon + Dv^\varepsilon|^2 - |P + Dv|^2) = \bar{H}^\varepsilon(P^\varepsilon) - \bar{H}(P) + \varepsilon \log \sigma^\varepsilon.$$

Since $|a|^2 - |b|^2 = -|a - b|^2 + 2a \cdot (a - b)$, we calculate that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{T}^n} (|P^\varepsilon + Dv^\varepsilon|^2 - |P + Dv|^2) \sigma^\varepsilon dx \\ &= -\frac{1}{2} \int_{\mathbb{T}^n} |(P^\varepsilon + Dv^\varepsilon) - (P + Dv)|^2 \sigma^\varepsilon dx \\ & \quad + \int_{\mathbb{T}^n} (P^\varepsilon + Dv^\varepsilon) \cdot ((P^\varepsilon + Dv^\varepsilon) - (P + Dv)) \sigma^\varepsilon dx \\ &= -\frac{1}{2} \int_{\mathbb{T}^n} |(P^\varepsilon + Dv^\varepsilon) - (P + Dv)|^2 \sigma^\varepsilon dx + (P^\varepsilon - P) \cdot \int_{\mathbb{T}^n} (P^\varepsilon + Dv^\varepsilon) \sigma^\varepsilon dx \\ &= -\frac{1}{2} \int_{\mathbb{T}^n} |(P^\varepsilon + Dv^\varepsilon) - (P + Dv)|^2 \sigma^\varepsilon dx + (P^\varepsilon - P) \cdot D\bar{H}^\varepsilon(P^\varepsilon), \end{aligned}$$

the second equality resulting from (1.7). Consequently (2.25) implies

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{T}^n} |(P^\varepsilon + Dv^\varepsilon) - (P + Dv)|^2 \sigma^\varepsilon dx &= \bar{H}(P) - \bar{H}^\varepsilon(P^\varepsilon) + (P^\varepsilon - P) \cdot D\bar{H}^\varepsilon(P^\varepsilon) \\ &\quad - \varepsilon \int_{\mathbb{T}^n} \log \sigma^\varepsilon \sigma^\varepsilon dx. \end{aligned}$$

The identity (2.23) follows, since $\bar{H}_\varepsilon^\varepsilon(P^\varepsilon) = - \int_{\mathbb{T}^n} \log \sigma^\varepsilon \sigma^\varepsilon dx$ according to (2.16).

2. To prove (2.24), we next integrate (2.25) with respect to the measure σ (recall (1.3)):

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{T}^n} (|P + Dv|^2 - |P^\varepsilon + Dv^\varepsilon|^2) d\sigma &= -\frac{1}{2} \int_{\mathbb{T}^n} |(P^\varepsilon + Dv^\varepsilon) - (P + Dv)|^2 d\sigma \\ &\quad + \int_{\mathbb{T}^n} (P + Dv) \cdot ((P + Dv) - (P^\varepsilon + Dv^\varepsilon)) d\sigma \\ &= -\frac{1}{2} \int_{\mathbb{T}^n} |(P^\varepsilon + Dv^\varepsilon) - (P + Dv)|^2 d\sigma + (P - P^\varepsilon) \cdot \int_{\mathbb{T}^n} P + Dv d\sigma \\ &= -\frac{1}{2} \int_{\mathbb{T}^n} |(P^\varepsilon + Dv^\varepsilon) - (P + Dv)|^2 d\sigma + (P^\varepsilon - P) \cdot D\bar{H}(P). \end{aligned}$$

Hence (2.25) gives

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{T}^n} |(P^\varepsilon + Dv^\varepsilon) - (P + Dv)|^2 d\sigma &= \bar{H}^\varepsilon(P^\varepsilon) - \bar{H}(P) + (P^\varepsilon - P) \cdot D\bar{H}(P) + \varepsilon \int_{\mathbb{T}^n} \log \sigma^\varepsilon d\sigma \\ &= \int_{\mathbb{T}^n} H^\varepsilon d\sigma - \bar{H}(P) + (P^\varepsilon - P) \cdot D\bar{H}(P). \end{aligned}$$

□

3 Linearizations and adjoints

3.1 Linearizing the PDE. The linearization about v^ε of the Euler-Lagrange equation (1.8) is the operator

$$(3.1) \quad L_\varepsilon[w] := -(P_i + v_{x_i}^\varepsilon)(P_j + v_{x_j}^\varepsilon)w_{x_i x_j} - (2(P_j + v_{x_j}^\varepsilon)v_{x_i x_j}^\varepsilon + W_{x_i})w_{x_i} - \varepsilon \Delta w,$$

defined for smooth, periodic functions $w : \mathbb{T}^n \rightarrow \mathbb{R}$.

LEMMA 3.1 *We have the alternative formulas*

$$(3.2) \quad L_\varepsilon[w] = -(P_i + v_{x_i}^\varepsilon)((P_j + v_{x_j}^\varepsilon)w_{x_j})_{x_i} - H_{x_i}^\varepsilon w_{x_i} - \varepsilon \Delta w.$$

and

$$(3.3) \quad L_\varepsilon[w] = -\frac{1}{\sigma^\varepsilon} \left([(P_i + v_{x_i}^\varepsilon)(P_j + v_{x_j}^\varepsilon) + \varepsilon \delta_{ij}] \sigma^\varepsilon w_{x_j} \right)_{x_i}.$$

Proof. 1. Formula (3.2) follows immediately from (3.1).

2. Recall from (1.7) that $\operatorname{div}((P + Dv^\varepsilon)\sigma^\varepsilon) = 0$. Consequently the expression on the right hand side of (3.3) equals

$$-(P_i + v_{x_i}^\varepsilon)((P_j + v_{x_j}^\varepsilon)w_{x_j})_{x_i} - \varepsilon \frac{\sigma_{x_i}^\varepsilon}{\sigma^\varepsilon} w_{x_i} - \varepsilon \Delta w.$$

But this the formula (3.2) for $L_\varepsilon[w]$, since $\sigma^\varepsilon = e^{\frac{H^\varepsilon - \bar{H}^\varepsilon}{\varepsilon}}$. □

The linearization L_ε is useful, as it appears when we differentiate the nonlinear PDE (1.8):

THEOREM 3.2 *These identities hold:*

$$(3.4) \quad L_\varepsilon[v^\varepsilon] = -2(P_i + v_{x_i}^\varepsilon)v_{x_j}^\varepsilon v_{x_i x_j}^\varepsilon + P_i W_{x_i}.$$

$$(3.5) \quad L_\varepsilon[v_{x_k}^\varepsilon] = (P_i + v_{x_i}^\varepsilon)W_{x_i x_k} \quad (k = 1, \dots, n).$$

$$(3.6) \quad L_\varepsilon[v_{P_k}^\varepsilon] = 2(P_i + v_{x_i}^\varepsilon)v_{x_i x_k}^\varepsilon + W_{x_k} \quad (k = 1, \dots, n).$$

$$(3.7) \quad L_\varepsilon[x_k] = -2(P_i + v_{x_i}^\varepsilon)v_{x_i x_k}^\varepsilon - W_{x_k} \quad (k = 1, \dots, n).$$

$$(3.8) \quad L_\varepsilon[v_\varepsilon^\varepsilon] = \Delta v^\varepsilon.$$

$$(3.9) \quad L_\varepsilon[H^\varepsilon] = -|DH^\varepsilon|^2 - \varepsilon|D^2 v^\varepsilon|^2 - \varepsilon \Delta W.$$

Proof. 1. According to (3.1) and (1.8)

$$\begin{aligned} L_\varepsilon[v^\varepsilon] &= -(P_i + v_{x_i}^\varepsilon)(P_j + v_{x_j}^\varepsilon)v_{x_i x_j}^\varepsilon - \varepsilon \Delta v^\varepsilon - (2(P_j + v_{x_j}^\varepsilon)v_{x_i x_j}^\varepsilon + W_{x_i})v_{x_i}^\varepsilon \\ &= (P_i + v_{x_i}^\varepsilon)W_{x_i} - (2(P_j + v_{x_j}^\varepsilon)v_{x_i x_j}^\varepsilon + W_{x_i})v_{x_i}^\varepsilon \\ &= P_i W_{x_i} - 2(P_j + v_{x_j}^\varepsilon)v_{x_i x_j}^\varepsilon v_{x_i}^\varepsilon. \end{aligned}$$

This is (3.4).

2. We obtain (3.5) upon differentiating (1.8) with respect to x_k : the left hand side appears when the differentiation falls upon v^ε and the right hand side when the differentiation falls upon the term involving the potential W .

Similarly, (3.6) results from our differentiating (1.8) with respect to P_k , and (3.8) from our differentiating in ε . We directly compute from the definition (3.1) that (3.9) is also valid. \square

Remark. We observe from (3.6) and (3.7) that

$$(3.10) \quad L_\varepsilon[x_k + v_{P_k}^\varepsilon] = 0 \quad (k = 1, \dots, n).$$

But note also that $x + D_P v^\varepsilon$ is not \mathbb{T}^n -periodic. We will return to this point in Section 4.2. \square

3.2 The adjoint operator. We introduce next the adjoint L_ε^* of L_ε with respect to the standard inner product in $L^2(\mathbb{T}^n)$; so that

$$(3.11) \quad \int_{\mathbb{T}^n} L_\varepsilon[f]g \, dx = \int_{\mathbb{T}^n} f L_\varepsilon^*[g] \, dx$$

for all smooth, \mathbb{T}^n -periodic functions f and g .

THEOREM 3.3 (i) *We have*

$$(3.12) \quad L_\varepsilon^*[w] = - \left([(P_i + v_{x_i}^\varepsilon)(P_j + v_{x_j}^\varepsilon) + \varepsilon \delta_{ij}] \sigma^\varepsilon \left(\frac{w}{\sigma^\varepsilon} \right)_{x_i} \right)_{x_j}.$$

(ii) *Therefore*

$$(3.13) \quad L_\varepsilon^*[\sigma^\varepsilon w] = \sigma^\varepsilon L_\varepsilon[w]$$

and

$$(3.14) \quad L_\varepsilon^*[\sigma^\varepsilon] = 0.$$

Proof. The identity (3.12) follows from (3.3) and an integration by parts. \square

Remark. It follows from (3.13) (or (3.3)) that the operator L_ε , acting on smooth functions, is symmetric with respect to the L^2 inner product weighted by σ^ε :

$$(3.15) \quad \int_{\mathbb{T}^n} L_\varepsilon[f]g \sigma^\varepsilon dx = \int_{\mathbb{T}^n} f L_\varepsilon[g] \sigma^\varepsilon dx$$

for smooth, \mathbb{T}^n -periodic functions f, g . Perhaps the spectrum of L_ε contains useful dynamical information in the limit $\varepsilon \rightarrow 0$. \square

3.3 More identities. We can employ the foregoing formulas to rewrite some of the expressions from Section 2:

THEOREM 3.4 *We have the identity*

$$(3.16) \quad \varepsilon \operatorname{tr}(D^2 \bar{H}^\varepsilon(P)) = \varepsilon n + \int_{\mathbb{T}^n} (|P + Dv^\varepsilon - V^\varepsilon|^2 - |(P + Dv^\varepsilon) \cdot D_{xP}^2 v^\varepsilon|^2 - \varepsilon |D_{xP}^2 v^\varepsilon|^2) \sigma^\varepsilon dx$$

and consequently the estimate

$$(3.17) \quad \int_{\mathbb{T}^n} (|(P + Dv^\varepsilon) \cdot D_{xP}^2 v^\varepsilon|^2 + \varepsilon |D_{xP}^2 v^\varepsilon|^2) \sigma^\varepsilon dx \leq \varepsilon n + \int_{\mathbb{T}^n} |P + Dv^\varepsilon - V^\varepsilon|^2 \sigma^\varepsilon dx.$$

Proof. 1. Owing to (3.6), we have

$$L_\varepsilon[\tfrac{1}{2}|D_P v^\varepsilon|^2] = v_{P_k}^\varepsilon (2(H^\varepsilon)_{x_k} - W_{x_k}) - |(P + Dv^\varepsilon) \cdot D_{xP}^2 v^\varepsilon|^2 - \varepsilon |D_{xP}^2 v^\varepsilon|^2.$$

We multiply by σ^ε and integrate, recalling from (3.14) that $L_\varepsilon^*[\sigma^\varepsilon] = 0$:

$$(3.18) \quad \begin{aligned} & \int_{\mathbb{T}^n} (|(P + Dv^\varepsilon) \cdot D_{xP}^2 v^\varepsilon|^2 + \varepsilon |D_{xP}^2 v^\varepsilon|^2) \sigma^\varepsilon dx \\ &= \int_{\mathbb{T}^n} v_{P_k}^\varepsilon (2(P_i + v_{x_i}^\varepsilon) v_{x_i x_k}^\varepsilon + W_{x_k}) \sigma^\varepsilon dx \\ &= \int_{\mathbb{T}^n} v_{P_k}^\varepsilon ((P_i + v_{x_i}^\varepsilon) v_{x_i x_k}^\varepsilon + H_{x_k}^\varepsilon) \sigma^\varepsilon dx \\ &= \int_{\mathbb{T}^n} v_{P_k}^\varepsilon (P_i + v_{x_i}^\varepsilon) v_{x_i x_k}^\varepsilon \sigma^\varepsilon dx + \varepsilon \int_{\mathbb{T}^n} v_{P_k}^\varepsilon \sigma_{x_k}^\varepsilon dx \\ &= - \int_{\mathbb{T}^n} v_{P_k x_i}^\varepsilon (P_i + v_{x_i}^\varepsilon) v_{x_k}^\varepsilon \sigma^\varepsilon dx - \varepsilon \int_{\mathbb{T}^n} v_{P_k x_k}^\varepsilon \sigma^\varepsilon dx \\ &= - \int_{\mathbb{T}^n} v_{P_k x_i}^\varepsilon (P_i + v_{x_i}^\varepsilon) (P_k + v_{x_k}^\varepsilon - V_k^\varepsilon) \sigma^\varepsilon dx - \varepsilon \int_{\mathbb{T}^n} v_{P_k x_k}^\varepsilon \sigma^\varepsilon dx, \end{aligned}$$

the last equality following from (1.7).

2. In view of (2.15),

$$\begin{aligned} \varepsilon \operatorname{tr}(D^2 \bar{H}^\varepsilon(P)) &= \int_{\mathbb{T}^n} (\varepsilon |I + D_{x,P}^2 v^\varepsilon|^2 + |(P + Dv^\varepsilon) D_{x,P}^2 v^\varepsilon|^2 + |P + Dv^\varepsilon - V^\varepsilon|^2) \sigma^\varepsilon dx \\ &\quad + 2 \int_{\mathbb{T}^n} (P + Dv^\varepsilon) D_{x,P}^2 v^\varepsilon \cdot (P + Dv^\varepsilon - V^\varepsilon) \sigma^\varepsilon dx. \end{aligned}$$

We use (3.18) to see that the second term equals

$$-2 \int_{\mathbb{T}^n} (|(P + Dv^\varepsilon) \cdot D_{xP}^2 v^\varepsilon|^2 + \varepsilon |D_{xP}^2 v^\varepsilon|^2) \sigma^\varepsilon dx - 2\varepsilon \int_{\mathbb{T}^n} v_{P_k x_k}^\varepsilon \sigma^\varepsilon dx.$$

Therefore

$$\begin{aligned} \varepsilon \operatorname{tr}(D^2 \bar{H}^\varepsilon(P)) &= \int_{\mathbb{T}^n} (\varepsilon |I + D_{x,P}^2 v^\varepsilon|^2 + |(P + Dv^\varepsilon) D_{x,P}^2 v^\varepsilon|^2 + |P + Dv^\varepsilon - V^\varepsilon|^2) \sigma^\varepsilon dx \\ &\quad - 2 \int_{\mathbb{T}^n} (|(P + Dv^\varepsilon) \cdot D_{xP}^2 v^\varepsilon|^2 + \varepsilon |D_{xP}^2 v^\varepsilon|^2) \sigma^\varepsilon dx - 2\varepsilon \int_{\mathbb{T}^n} v_{P_k x_k}^\varepsilon \sigma^\varepsilon dx \\ &= \int_{\mathbb{T}^n} (|P + Dv^\varepsilon - V^\varepsilon|^2 - |(P + Dv^\varepsilon) D_{x,P}^2 v^\varepsilon|^2 + \varepsilon |I|^2 - \varepsilon |D_{x,P}^2 v^\varepsilon|^2) \sigma^\varepsilon dx. \end{aligned}$$

The formula (3.16) follows; as does the inequality (3.17), since $\operatorname{tr}(D^2 \bar{H}^\varepsilon(P)) \geq 0$. \square

THEOREM 3.5 *We have*

$$(3.19) \quad \varepsilon \bar{H}_{\varepsilon\varepsilon}^\varepsilon(P) = \int_{\mathbb{T}^n} \left(\frac{|H^\varepsilon - \bar{H}^\varepsilon(P) + \varepsilon \bar{H}_\varepsilon^\varepsilon(P)|^2}{\varepsilon^2} - |(P + Dv^\varepsilon) \cdot Dv_\varepsilon^\varepsilon|^2 - \varepsilon |Dv_\varepsilon^\varepsilon|^2 \right) \sigma^\varepsilon dx,$$

and therefore

$$(3.20) \quad \int_{\mathbb{T}^n} (|(P + Dv^\varepsilon) \cdot Dv_\varepsilon^\varepsilon|^2 + \varepsilon |Dv_\varepsilon^\varepsilon|^2) \sigma^\varepsilon dx \leq \int_{\mathbb{T}^n} \frac{|H^\varepsilon - \bar{H}^\varepsilon(P) + \varepsilon \bar{H}_\varepsilon^\varepsilon(P)|^2}{\varepsilon^2} \sigma^\varepsilon dx.$$

Proof. 1. According to (3.8), we have

$$L_\varepsilon \left[\frac{1}{2} (v_\varepsilon^\varepsilon)^2 \right] = v_\varepsilon^\varepsilon \Delta v^\varepsilon - |(P + Dv^\varepsilon) \cdot Dv_\varepsilon^\varepsilon|^2 - \varepsilon |Dv_\varepsilon^\varepsilon|^2.$$

We multiply by σ^ε and integrate:

$$(3.21) \quad \int_{\mathbb{T}^n} (|(P + Dv^\varepsilon) \cdot Dv_\varepsilon^\varepsilon|^2 + \varepsilon |Dv_\varepsilon^\varepsilon|^2) \sigma^\varepsilon dx = \int_{\mathbb{T}^n} v_\varepsilon^\varepsilon \Delta v^\varepsilon \sigma^\varepsilon dx.$$

The formula (2.18) implies

$$\begin{aligned} \varepsilon \bar{H}_{\varepsilon\varepsilon}^\varepsilon(P) &= \int_{\mathbb{T}^n} \left(\varepsilon |Dv_\varepsilon^\varepsilon|^2 + |(P + Dv^\varepsilon) \cdot Dv_\varepsilon^\varepsilon|^2 + \frac{|H^\varepsilon - \bar{H}^\varepsilon(P) + \varepsilon \bar{H}_\varepsilon^\varepsilon(P)|^2}{\varepsilon^2} \right) \sigma^\varepsilon dx \\ &\quad - 2 \int_{\mathbb{T}^n} (P + Dv^\varepsilon) \cdot Dv_\varepsilon^\varepsilon \frac{H^\varepsilon - \bar{H}^\varepsilon(P) + \varepsilon \bar{H}_\varepsilon^\varepsilon(P)}{\varepsilon} \sigma^\varepsilon dx. \end{aligned}$$

Recalling yet again (1.7), we observe that the second integral term equals

$$2 \int_{\mathbb{T}^n} v_\varepsilon^\varepsilon (P + Dv^\varepsilon) \cdot \frac{DH^\varepsilon}{\varepsilon} \sigma^\varepsilon dx = -2 \int_{\mathbb{T}^n} v_\varepsilon^\varepsilon \Delta v^\varepsilon \sigma^\varepsilon dx,$$

the last equality following from (1.8). We substitute (3.21) and rewrite, obtaining (3.19). \square

4 Some applications

We collect in the concluding section some applications of the foregoing formulas, of which those in Section 4.2 concerning nonresonance are the most interesting.

4.1 \bar{H}^ε as $\varepsilon \rightarrow 0$. An overall goal is understanding how \bar{H} and its approximations \bar{H}^ε for small $\varepsilon > 0$ provide analytic control of v^ε , σ^ε , and thus in the limit of v , σ .

As an illustration, we show next that if $\bar{H}^\varepsilon(P)$ is nice enough as a function of ε near zero, then we can construct a limit measure σ that is absolutely continuous with respect to Lebesgue measure:

THEOREM 4.1 *If*

$$(4.1) \quad \left(\frac{\bar{H}_{\varepsilon\varepsilon}^\varepsilon(P)}{\varepsilon} \right)^{1/2} \text{ is integrable near } \varepsilon = 0,$$

then

$$(4.2) \quad \sigma^\varepsilon \rightarrow \sigma \quad \text{in } L^1(\mathbb{T}^n)$$

and $\sigma \in L^1(\mathbb{T}^n)$ *solves* (1.3).

Proof. If $0 < \varepsilon_1 < \varepsilon_2$, we have

$$\begin{aligned} \int_{\mathbb{T}^n} |\sigma^{\varepsilon_2} - \sigma^{\varepsilon_1}| dx &\leq \int_{\varepsilon_1}^{\varepsilon_2} \int_{\mathbb{T}^n} |\sigma_\varepsilon^\varepsilon| dx d\varepsilon \\ &\leq \int_{\varepsilon_1}^{\varepsilon_2} \left(\int_{\mathbb{T}^n} \frac{|\sigma_\varepsilon^\varepsilon|^2}{\sigma^\varepsilon} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^n} \sigma^\varepsilon dx \right)^{\frac{1}{2}} d\varepsilon \leq \int_{\varepsilon_1}^{\varepsilon_2} \left(\frac{\bar{H}_{\varepsilon\varepsilon}^\varepsilon}{\varepsilon} \right)^{\frac{1}{2}} d\varepsilon, \end{aligned}$$

according to (2.18). Consequently, (4.1) implies that $\{\sigma^\varepsilon\}_{\varepsilon>0}$ is a Cauchy sequence in $L^1(\mathbb{T}^n)$ as $\varepsilon \rightarrow 0$. \square

4.2 Nonresonance phenomena. We assume hereafter that we can select P^ε so that

$$(4.3) \quad V := D\bar{H}^\varepsilon(P^\varepsilon)$$

does not depend upon ε . Write $V = (V_1, \dots, V_n)$. We suppose also the *nonresonance condition* that for some constant $c > 0$

$$(4.4) \quad |V \cdot k| \geq \frac{c}{|k|^\gamma} \quad \text{for all vectors } k \in \mathbb{Z}^n, k \neq 0.$$

Next, take $g : \mathbb{T}^n \rightarrow \mathbb{R}$ to be smooth and have zero mean:

$$\int_{\mathbb{T}^n} g(X) dX = 0.$$

Then using a standard Fourier series representation and the nonresonance condition (4.4), we have

LEMMA 4.2 *There exists a smooth \mathbb{T}^n -periodic solution $f = f(X)$ of the linear elliptic PDE*

$$(4.5) \quad -V_k V_l f_{X_k X_l} = g \quad \text{in } \mathbb{T}^n.$$

Furthermore, we have for each $s \geq 0$ the estimate

$$(4.6) \quad \|f\|_{H^s(\mathbb{T}^n)} \leq C_s \|g\|_{H^{s+\gamma}(\mathbb{T}^n)},$$

for a constant C_s .

THEOREM 4.3 *Assume that*

$$(4.7) \quad \varepsilon \operatorname{tr}(D^2 \bar{H}^\varepsilon(P^\varepsilon)) = o(1) \quad \text{as } \varepsilon \rightarrow 0.$$

Then for each smooth function $g : \mathbb{T}^n \rightarrow \mathbb{R}$ we have

$$(4.8) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}^n} g(x + D_P v^\varepsilon) \sigma^\varepsilon dx = \int_{\mathbb{T}^n} g(X) dX.$$

Remark. This is a variant of a theorem in [E1]. The formal interpretation is that under the symplectic change of variable

$$(x, p) \rightarrow (X, P),$$

defined implicitly by the formulas $p = P + D_x v$, $X = x + D_P v$ the dynamics become linear: $X(t) = X_0 + tV$ for $t \geq 0$. Since $V \cdot k \neq 0$ for all $k \in \mathbb{Z}^n - \{0\}$, the flow is therefore asymptotically equidistributed with respect Lebesgue measure. The rigorous assertion (4.8) is consistent with this picture. \square

Proof. 1. Subtracting a constant if necessary, we may assume the average of g is zero. Now let f solve the linear PDE (4.5), and define

$$\tilde{f}(x) := f(x + D_P v^\varepsilon).$$

The function \tilde{f} is \mathbb{T}^n -periodic, although $x + D_P v^\varepsilon$ is not.

Recalling from (3.10) that $L_\varepsilon[x + D_P v^\varepsilon] = 0$, we compute

$$\begin{aligned} L_\varepsilon[\tilde{f}] &= D_X f \cdot L_\varepsilon[x + D_P v^\varepsilon] \\ &\quad - [(P_i + v_{x_i}^\varepsilon)(P_j + v_{x_j}^\varepsilon) + \varepsilon \delta_{ij}](\delta_{ik} + v_{x_i P_k}^\varepsilon)(\delta_{jl} + v_{x_j P_l}^\varepsilon) f_{X_k X_l} \\ &= -[(P_i + v_{x_i}^\varepsilon)(P_j + v_{x_j}^\varepsilon) + \varepsilon \delta_{ij}](\delta_{ik} + v_{x_i P_k}^\varepsilon)(\delta_{jl} + v_{x_j P_l}^\varepsilon) f_{X_k X_l}. \end{aligned}$$

Here and afterwards f is evaluated at $x + D_P v^\varepsilon$. It follows that

$$(4.9) \quad L_\varepsilon[\tilde{f}] + g(x + D_P v^\varepsilon) = E_1 + E_2,$$

for

$$\begin{aligned} E_1 &:= [(P_i + v_{x_i}^\varepsilon)(\delta_{ik} + v_{x_i P_k}^\varepsilon)(P_j + v_{x_j}^\varepsilon)(\delta_{jl} + v_{x_j P_l}^\varepsilon) - V_k V_l] f_{X_k X_l}, \\ E_2 &:= \varepsilon(\delta_{ik} + v_{x_i P_k}^\varepsilon)(\delta_{il} + v_{x_i P_l}^\varepsilon) f_{X_k X_l} \end{aligned}$$

2. Selecting s large enough, we deduce from (4.6) that $\|f\|_{C^{1,1}}$ is bounded. Consequently, (2.15) implies the estimate

$$\begin{aligned} \int_{\mathbb{T}^n} |E_1| \sigma^\varepsilon dx &\leq C \int_{\mathbb{T}^n} (1 + |I + D_{x,P}^2 v^\varepsilon|) |(P + Dv^\varepsilon)(I + D_{x,P}^2 v^\varepsilon) - V^\varepsilon| \sigma^\varepsilon dx \\ &\leq C(\varepsilon \operatorname{tr}(D^2 \bar{H}^\varepsilon(P^\varepsilon)))^{\frac{1}{2}} = o(1). \end{aligned}$$

Likewise,

$$\int_{\mathbb{T}^n} |E_2| \sigma^\varepsilon dx \leq C\varepsilon \operatorname{tr}(D^2 \bar{H}^\varepsilon(P^\varepsilon)) = o(1).$$

3. It follows now from (4.9) and (3.14) that

$$\begin{aligned} \int_{\mathbb{T}^n} g(x + D_P v^\varepsilon) \sigma^\varepsilon dx &= \int_{\mathbb{T}^n} (-L_\varepsilon[\tilde{f}] - E_1 - E_2) \sigma^\varepsilon dx \\ &= - \int_{\mathbb{T}^n} \tilde{f} L_\varepsilon^*[\sigma^\varepsilon] dx + o(1) \\ &= o(1) \end{aligned}$$

as $\varepsilon \rightarrow 0$. □

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