MATH 10B – METHODS OF MATHEMATICS: CALCULUS, STATISTICS AND COMBINATORICS

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Topics in Mathematics 10B

- Part 1: COMBINATORICS
- Part 2: DISCRETE PROBABILITY THEORY
- Part 3: DYNAMICS
- Part 4: MATRIX ALGEBRA

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Overview of Part 1: Combinatorics

Combinatorics is the study of arrangements of objects and their enumeration, and in particular **the counting of objects with certain properties**.

We can phrase many counting problems in terms of counting how many ordered or unordered arrangements of the objects of a set.

Simple examples include counting the number of

- different telephone numbers possible in the US,
- allowable passwords on a computer system, and
- different orders in which runners in a race can finish.

1. Product and Sum Rules

2. Inclusion-Exclusion Principle

3. Pigeonhole Principle

4. Permutations and Combinations

5. Binomial Coefficients

6. More Counting: Balls into Boxes

7. Algorithms

Section 1

Product and Sum Rules

A. The Product Rule

Notation

If A is a finite set, we write

$$|A|$$

to denote the number of elements in A.

Example 1.1

If $A_1, A_2, ..., A_m$ are finite sets, their **Cartesian product** *is the set*

$$A_1 \times A_2 \times \cdots \times A_m = \{(a_1, \ldots, a_m) \mid a_k \in A_k \;\; k = 1, \ldots m\}.$$

The number of elements in the Cartesian product of these sets is the product of the number of elements in each set; that is,

$$\boxed{|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdots |A_m|.}$$

$\square$

We can generalize the preceding example into a counting principle:

THEOREM (Product Rule)

Suppose that a procedure can be broken down into a sequence of tasks, $T_1, T_2, ..., T_m$.
If each task T_i *can be done in* n_i *ways, regardless of how the previous tasks were done, then there are*

$$\boxed{n_1 \cdot n_2 \cdots n_m}$$

ways to carry out the procedure.

Example 1.2

The chairs of an auditorium are to be labeled with a letter and with a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

- *Task 1: assign each chair one of the 26 letters*
- *Task 2: assign each chair one of the 100 possible integers*

There are $26 \cdot 100 = 2600$ *different ways a chair can be labeled.* $\square$

Example 1.3

How many different license plates are available if each plate contains a sequence of three letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene).

- *Task 1: assign the first space one of the 26 letters*
- *Task 2: assign the second space one of the 26 letters*
- *Task 3: assign the third space one of the 26 letters*
- *Task 4: assign the fourth space one of the 10 possible digits*
- *Task 5: assign the fifth space one of the 10 possible digits*
- *Task 6: assign the sixth space one of the 10 possible digits*

There are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 =$ *17,576,000 possible license plates.* $\square$

THEOREM

The number of different subsets of a finite set S is

$$\boxed{2^{|S|}.}$$

Proof.

Let S be a finite set. List the elements of S in arbitrary order. Imagine each of these elements could be in the subsets or not in the subset. For each element, there are 2 options. Thus there are $2^{|S|}$ different subsets, and this includes both the empty set $\emptyset$ and the entire set S. $\square$

B. The Sum Rule

Example 1.4

If $A_1, A_2, ..., A_m$ are **disjoint** *finite sets, then the number of elements in the union of these sets is the sum of the number of elements in each set:*

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

.

□

(We will later see how to modify this formula if the sets $A_1, A_2, ..., A_m$ are not disjoint.)

We can generalize the foregoing example into a general counting principle:

THEOREM (Sum Rule)

Assume that task can be done either in one of n_1 ways, in one of n_2 ways, ..., or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the n_j ways for $1 \leq i < j \leq m$.

Then the number of ways to do the task is

$$\boxed{n_1 + n_2 + \cdots + n_m.}$$

Example 1.5

Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee, How many different choices are there for this representative if there are 37 mathematics faculty and 83 mathematics majors and no one is both a faculty and a student?

- *There are 37 ways to choose a math faculty.*
- *There are 83 ways to choose a math major.*

Then there are $37 + 83 = 120$ *ways to pick this representative.* □

Example 1.6

Suppose we are throwing two dice and we win a dollar if the sum is less than or equal to 3. How many ways are there to win?

- *There is 1 way to get a sum = 2. (1 dot on each)*
- *There are 2 ways get a sum = 3 (1 dot on one and 2 dots on the other).*

So there are in all 3 *ways to win.* □

Example 1.7

Suppose a phenotype is expressed when the genotype is either homozygous (AA) or heterozygous (Aa or aA).

- *There is 1 way to be homozygous AA.*
- *There are 2 ways to be heterozygous.*

There are 3 *ways to have a specific phenotype.* □

Example 1.8

You need to choose a password, which is 6 to 8 characters long, where each character is a lowercase letter or a digit. How many possible passwords are there?

- *Let* P = *total number of passwords and* P_6, P_7, P_8 *be the number of passwords with 6, 7, 8 characters respectively.*
- *Sum Rule:* $P = P_6 + P_7 + P_8$
- *Product Rule:* $P_6 = 36 \cdot 36 \cdots 36 = 36^6$ *(26+10 choices for each character)*
- *Similarly,* $P_7 = 36^7$ *and* $P_8 = 36^8$

So $P = 36^6 + 36^7 + 36^8$

Example 1.9

Same question as before, but now let's count passwords which have at least one digit.

- *Need to subtract from P_6, P_7, P_8, all the passwords with no digits. This is simply using the fact that $A = 1 - A^c$*
- *Number of passwords with with no digits for P_6, P_7, P_8 respectively are 26^6, 26^7 and 26^8*

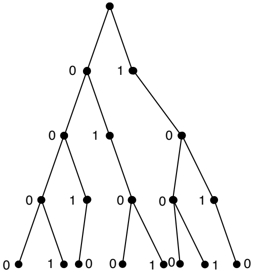
Thus $P = (36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8)$

C. More Counting: Tree diagrams

A visual that can help solve counting problems is a **tree diagram**.

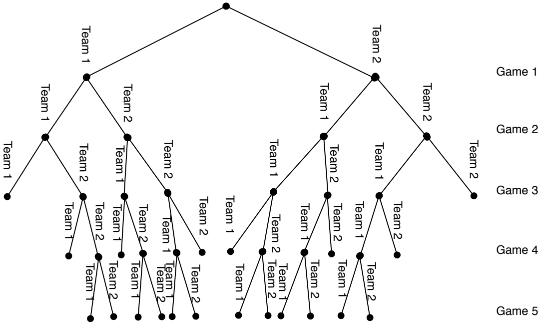
A **tree** consists of a **root**, a number of **branches** leaving the root, and possible additional branches leaving the endpoints of other branches. A **leaf** of the tree is an endpoint of a branch from which no other branches begin.

To use the tree in counting, we use a branch to represent each possible choice. The possible outcomes are then represented by the leaves.



Example 1.10

A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?



We count the leaves to find that there are 20 different possibilities.

Section 2

Inclusion-Exclusion Principle

Suppose that a task can be done in n_1 or in n_2 ways, but that some of the set of n_1 ways to do the task are the same as some of the n_2 ways to do the task. We cannot just use the Sum Rule here because it will lead to an overcount. Instead, we subtract out the number of ways n_1 and n_2 are in common.

THEOREM (Inclusion-Exclusion Principle)

If A_1 and A_2 are finite sets, then the number of elements in the union of these sets is the sum of the number of elements in each set minus the number of elements in both sets.

Therefore

$$\boxed{|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.}$$

Example 2.1

A computer company receives 350 applicants from new graduates. Suppose that 220 of these majored in computer science, 147 majored in business, and 51 majored in both. How many of these applicants majored neither in CS or business?

- *Total number who majored in either CS or business:* $220 + 147 - 51 = 316$
- *Total number who did not major in CS or business:* $350 - 316 = 34$.

$\square$

We extend the previous counting principe to finitely many finite sets:

THEOREM (General Inclusion-Exclusion Principle)

We have

$$\boxed{\begin{aligned}\left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| \\ &\quad + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| \\ &\quad - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.\end{aligned}}$$

We will prove the theorem by **induction**, which is a common type of proof in discrete mathematics. There are two steps:

(1) **Prove that the statement is true for $n = 1$.**
(2) **Prove that if the statement is true for some value n, then it must also be true for $n + 1$.**

This is enough to prove the statement for all n. Notice that if we prove (1), then (2) implies that the statement holds for $n = 2$. Of course, we can repeat step (2) as many times as we like to prove the the statement is true for *any* value of n.

Proof.

Step (1) is easy, since when n=1, the statement reads

$$|A_1| = |A_1|$$

This obviously being true, we move on to step (2).

Suppose next that the Theorem holds for some particular choice of n: this is the **induction hypothesis**. We want to use this to find the corresponding formula with $n+1$ replacing n.

First note that $\bigcup_{i=1}^{n+1} A_i = \left(\bigcup_{i=1}^{n} A_i\right) \bigcup A_{n+1}$. So we can rewrite, using our rule for the union of two events:

$$
\begin{aligned}
\left|\bigcup_{i=1}^{n+1} A_i\right| &= \left|\bigcup_{i=1}^{n} A_i \bigcup A_{n+1}\right| \\
&= \left|\bigcup_{i=1}^{n} A_i\right| + |A_{n+1}| - \left|\left(\bigcup_{i=1}^{n} A_i\right) \bigcap A_{n+1}\right| \\
&= \left|\bigcup_{i=1}^{n} A_i\right| + |A_{n+1}| - \left|\bigcup_{i=1}^{n} \left(A_i \bigcap A_{n+1}\right)\right|
\end{aligned}
$$

Now use the induction hypothesis to rewrite the first and third terms:

$$
\begin{aligned}
\left|\bigcup_{i=1}^{n+1} A_i\right| &= \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| \\
&\quad - \cdots + (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n| \\
&\quad + |A_{n+1}| \\
&\quad - \Big[\sum_{i=1}^{n} |A_i \cap A_{n+1}| - \sum_{1\le i<j\le n} |A_i \cap A_j \cap A_{n+1}| \\
&\quad + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k \cap A_{n+1}| \\
&\quad - \cdots + (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}| \Big]
\end{aligned}
$$

Rearranging (left as an exercise), we have

$$\begin{aligned}\left|\bigcup_{i=1}^{n+1} A_i\right| = \sum_{i=1}^{n+1} |A_i| - \sum_{1\le i<j\le n+1} |A_i \cap A_j| + \\ \sum_{1\le i<j<k\le n+1} |A_i \cap A_j \cap A_k| \\ - \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_{n+1}|.\end{aligned}$$

This is precisely the expression that we would generate from the formula for the union of $n+1$ events.

Thus, if the statement holds for n events, it holds for $n+1$ events. By induction, the Theorem is proved. $\square$

Section 3

Pigeonhole Principle

THEOREM (Pigeonhole Principle)

If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

This assertion is of course obvious, but we will give a formal proof to illustrate the mathematical idea of the contrapositive.

Notation

The **contrapositive** *of a conditional statement*

If P, then Q

is the statement

If not Q, then not P.

If the original statement is true, then the contrapositive is true, and vice-versa.

Proof.

First, we assume that none of the k boxes contains more than one object (the opposite of 'at least one box contains two or more'). Now, we will show that the number of objects placed into k boxes has to be less than $k + 1$. If no more than one object is in a box, then the total number of objects would be at most k. We have shown the contrapositive. □

Example 3.1

- *Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.*
- *If an exam is graded on a scale from 0 to 100 points, then among 102 students, there must be at least 2 students with the same score.*

□

Notation

Let x be a real number.

(i) *The largest integer less than or equal to x is denoted*

$$\lfloor x \rfloor.$$

We sometimes call $\lfloor x \rfloor$ the **floor** *of x.*

(ii) *The smallest integer greater than or equal to x is denoted*

$$\lceil x \rceil.$$

We sometimes refer to $\lceil x \rceil$ as the **ceiling** *of x.*

Note that

$$\lfloor x \rfloor \le x \le \lceil x \rceil$$

and

$$\lfloor x \rfloor \le x \le \lfloor x \rfloor + 1, \quad \lceil x \rceil - 1 \le x \le \lceil x \rceil.$$

THEOREM (Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

We will use a proof by contradiction. First, we assume that the statement is not true, but rather the opposite is true. Then we will find a contradiction.

Proof.

Suppose that none of the boxes contains more than $\lceil N/k \rceil$-1 objects. Then, the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N$$

where the inequality $\lceil N/k \rceil < (N/k) + 1$ has been used. This is a contradiction because there are a total of N objects. $\square$

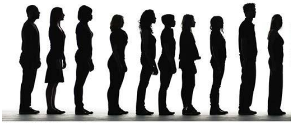
Example 3.2

- *Among 100 people, there are at least* $\lceil 100/12 \rceil = 9$ *who were born in the same month.*
- *9 cards must be selected from a standard deck of 52 cards to guarantee that at least 3 cards of the same suit are chosen.*
  - *Choose smallest integer* N *such that* $\lceil N/4 \rceil \geq 3$*. So,* $N/4 + 1 > \lceil N/4 \rceil \geq 3$ *and* $N > 8$*.*
- *If there are 30 students in a class, then at least* $\lceil 30/26 \rceil = 2$ *have last names that begin with the same letter.*

$\square$

Example 3.3

Ten people of different heights line up in a row. Show that it is always possible to select four of them to step sideways to form a shorter row in which their heights from left to right are either increasing or decreasing.



42-22342758 [RF] © www.visualphotos.com

SOLUTION. *Let* a_k *denote the height of the* k*-th person from the left, for* $k = 1, \ldots, 10$*. We will show that we can select subindices*

$$k_1 < k_2 < k_3 < k_4$$

from among $\{1, 2, \ldots, 10\}$ *such that either*

$$a_{k_1} < a_{k_2} < a_{k_3} < a_{k_4} \quad \text{(Case 1: increasing heights)}$$

or else

$$a_{k_1} > a_{k_2} > a_{k_3} > a_{k_4} \quad \text{(Case 2: decreasing heights).}$$

To see this, assume that there do not exist 4 subindices giving increasing heights (Case 1). We will show we can then find 4 subindices giving Case 2.

Define m_k to be the length of the longest increasing subsequence of increasing heights, starting at position k. Then $1 \leq m_k \leq 3$, since Case 1 never holds. So we have 10 integers $\{m_1, m_2, \ldots, m_{10}\}$, each of which is $1, 2$ or 3. Consequently the Extended Pigeonhole Principle implies that at least 4 of them are equal, say

$$m_{k_1} = m_{k_2} = m_{k_3} = m_{k_4}.$$

But this implies that Case 2 is valid for these indices. To see this, note for instance that $m_{k_1} = m_{k_2}$ implies $a_{k_1} > a_{k_2}$, since otherwise m_{k_1} would be strictly greater than m_{k_2}. □

Section 4

Permutations and Combinations

Introduction

- Many problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, **where the order of the elements matters**. Such arrangements are called **permutations**.

  For example, in how many ways can we select three students from a group of five students to stand in line for a picture?

- Many other problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, **where the order of the elements does not matter**. These sort of arrangements are called **combinations**.

  For example, how many different committees of three students can be formed from a group of four students?

A. Permutations

DEFINITION

(i) *A* **permutation** *of a set of distinct objects is an* **ordered** *arrangement of these objects.*

(ii) *An ordered arrangement of r elements of a set is called an* **r-permutation**.

Example 4.1

Let $A = \{1, 2, 3\}$. *The ordered arrangement 3, 1, 2 is a permutation of A. The ordered arrangement 3, 2 is a 2-permutation of A.*
In how many ways can we select three students from a group of five students to stand in line for a picture?

- *There are 5 ways to select the first student in line.*
- *There are 4 ways to select the second student in line.*
- *There are 3 ways to select the third student in line.*

So there are $5 \cdot 4 \cdot 3 = 60$ *ways*

Example 4.2

How many permutations are there of an n element set?
We want to count the number of ways of putting n things in order:

- *First, we must pick one thing to be the first in the list. There are n ways of doing this.*
- *Next, we must pick the second thing. Since we used one object already, there are $n-1$ ways of doing this.*
- *Next, we pick the third thing. There are $n-2$ ways of doing this.*
- *And so on...*
- *Finally, the pick the last element in the list. There is only 1 way to do this.*

Therefore the Product Rule implies that there are

$$\boxed{n(n-1)(n-2)\cdots 2\cdot 1 = n!}$$

permutations of a set with n elements.

DEFINITION

When n is a nonnegative integer, we define the **factorial** *of n to be*

$$\boxed{n! = \begin{cases} n(n-1)(n-2)\cdots 2\cdot 1 & \textit{if } n \geq 1 \\ 0 & \textit{if } n = 0 \end{cases}}$$

Example 4.3

The set $\{A, B, C, D, E\}$ *has 5 elements and thus there are* $5! = 120$ *permutations of it.*

□

Example 4.4

There are $52!$ *permutations of a standard deck of cards.* $52! \approx 8 \times 10^{67}$, *an extraordinarily huge number. For reference, a 59-card deck would have* $59!$ *permutations, which is more than the number of atoms in the universe (*$\approx 10^{80}$*).*

□

THEOREM

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$\boxed{P(n,r) = n(n-1)(n-2)\cdots(n-r+1)}$$

r-permutations of a set with n distinct elements.

Proof.

The first element of the permutation can be chosen in n ways because there are n elements in the set. There are $n-1$ ways to choose the second element of the permutation since there are $n-1$ elements left. Similarly, there are $n-2$ ways to choose the third element and so on, until there are exactly $n-(r-1) = n-r+1$ ways to choose the rth element. By the Product Rule, there are

$$n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of the set. □

THEOREM

If n and r are integers with $0 \leq r \leq n$, then

$$\boxed{P(n, r) = \frac{n!}{(n-r)!}}$$

Proof.

Simply write out the product (by definition of the factorial) and cancel terms. □

Example 4.5

How many permutations of the letters ABCDEFGH contain the string ABC?

- *The letters ABC must occur as a block.*
- *Find the number of permutations of six objects (ABC, D, E, F, G, H)*
- $\frac{6!}{0!} = 6! = 720$ *permutations*

Example 4.6

How many words with three letters can be made from the English alphabet with no repetition allowed?

Answer: A three letter word is just another name for the 3-permutation of letters. There are 26 letters in the alphabet, so there are $26 \cdot 25 \cdot 24 = 15,600$ *possible three letter words without any repeated letters.*

B. Combinations

DEFINITION

A **r-combination** *of elements of a set is an* **unordered** *selection of r elements from the set.*

In other words, an r-combination is simply a subset of the set with r elements.

Example 4.7

Let $A = \{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a 3-combination from A.

THEOREM

Let n be a nonnegative integer and r an integer with $0 \leq r \leq n$. The number of r-combinations of a set with n elements is

$$\boxed{C(n, r) = \frac{n!}{r!(n-r)!}}$$

Proof.

The r-permutations of the set can be obtained by forming the $C(n,r)$ r-combinations of the set, and then ordering the elements in each r-combination, which can be done in $P(r,r)$ ways. Consequently,

$$P(n,r) = C(n,r) \cdot P(r,r)$$

This implies that

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!(r-r)!} = \frac{n!}{r!(n-r)!}$$

$\square$

Note that $C(n,r) = \frac{n(n-1)\cdots(n-r+1)}{r!}$.

Example 4.8

- *There are* $C(52,5) = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 2,598,960$ *possible poker hands of five cards that can be dealt from a standard deck of 52 cards.*
- *There are* $C(52,47) = \frac{52!}{47!5!} = 2,598,960$ *different ways of selecting 47 cards from a standard deck of 52 cards.*

THEOREM

Let n *and* r *be nonnegative integers with* $r \leq n$. *Then*

$$\boxed{C(n,r) = C(n,n-r)}.$$

We will provide a combinatorial proof.

A **combinatorial proof** of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects, but in different ways.

Proof.

Let S be a set with n elements. LHS (left hand side) counts k-element subsets of S. RHS (right hand side) counts $(n-k)$-element subsets of S. Are these numbers the same? Yes, because there is a bijection between k-element subsets and $(n-k)$ element subsets that maps a set A to $S - A$. $\square$

Example 4.9

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

- *The number of 5-combinations of a set with 10 elements.*
- $C(10, 5) = \frac{10!}{5!5!} = 252$ *number of ways*

Section 5

Binomial Coefficients

A. Binomial Theorem

Notation

We often denote the number of r-combinations from a set with n elements by

$$C(n,r) = \boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!}.}$$

We read this as "n choose r."

We also call $\binom{n}{r}$ a **binomial coefficient**, as it appears in the following expansion, known as the **Binomial Theorem**:

THEOREM

Let x and y be variables, and let n be a nonnegative integer. Then

$$\boxed{(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j}$$

Proof.

The terms in the product when it is expanded are of the form $x^{n-j}y^j$ for $j = 0, 1, 2, ..., n$.

$$
\begin{aligned}
(x+y)^n &= \overbrace{(x+y)(x+y)\cdots(x+y)}^{n} \\
&= \alpha_0 x^n + \alpha_1 x^{n-1} y + \alpha_2 x^{n-2} y^2 + \cdots + \alpha_{n-1} x y^{n-1} + \alpha_n y^n
\end{aligned}
$$

To count the number of terms of the form $x^{n-j}y^j$, note that to obtain such a term it is necessary to choose $n-j$ x's from the n sums of $(x+y)$ (so that the other j terms in the product are y's). Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$, which is equal to $\binom{n}{j}$. $\square$

Example 5.1

What is the coefficient of x^3y^4 in the expansion of $(x+y)^7$?

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35$$

THEOREM

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof.

1. First proof: Using the Binomial Theorem with $x = 1$ and $y = 1$, we see that

$$2^n = (1+1)^n = \sum_{k=1}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=1}^{n} \binom{n}{k}.$$

Second proof: The left hand side is the sum on k of the number of subsets of size k of n-element set. The right hand side counts the number of subsets of n-element set. □

THEOREM

Let n be a nonnegative integer. Then we have the identities

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0, \quad \sum_{k=0}^{n} 2^k\binom{n}{k} = 3^n$$

Proof.

We have

$$0 = (1-1)^n = \sum_{k=1}^{n}\binom{n}{k}(-1)^k 1^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}.$$

and

$$3^n = (2+1)^n = \sum_{k=1}^{n}\binom{n}{k}2^k 1^{n-k} = \sum_{k=0}^{n}2^k\binom{n}{k}.$$

$\square$

THEOREM

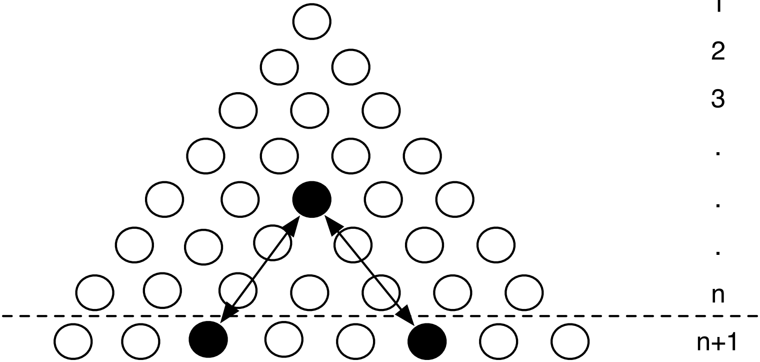
Let n be a nonnegative integer. Then

$$\sum_{k=1}^{n} k = \binom{n+1}{2}$$

Proof.

We will use a visual proof to show that there exists a bijection between a set of $\sum_{k=1}^{n} k$ objects and the set of 2-element subsets of a set with $n+1$ objects.

$\square$



$\sum_{k=1}^{n} k$ is the total number of balls above the line. You can identify any ball above the line by choosing two balls below the line that make up the bottom of a equilateral triangle.

B. Pascal's Triangle

Pascal's triangle is a triangular array of integers, the first 4 rows of which are:

$$
\begin{array}{lccccccc}
n=0: & & & & 1 & & & \\
n=1: & & & 1 & & 1 & & \\
n=2: & & 1 & & 2 & & 1 & \\
n=3: & 1 & & 3 & & 3 & & 1
\end{array}
$$

We compute the entries of each row by adding the two entries in the previous row, to the upper left and upper right of the current position. So the $n = 4$ row reads

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

and the $n = 5$ row is

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1.$$

The next theorem shows that in working out Pascal's triangle, we are actually computing binomial coefficients.

THEOREM

Let n and k be positive integers with $n \geq k$. Then

$$\boxed{\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}}$$

We will give a combinatorial proof.

Proof.

Suppose that T is a set containing $n+1$ elements. Let a be an element in T, and let $S = T - \{a\}$. Note that there are $\binom{n+1}{k}$ subsets of T containing k elements. However, a subset of T with k elements either contains a together with $k-1$ elements of S, or contains k elements of S and does not contain a.

Because there are $\binom{n}{k-1}$ subsets of $k-1$ elements of S, there are $\binom{n}{k-1}$ subsets of k elements of T that contain a. And there are $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S. Consequently, the Sum Rule gives the formula on the previous page. $\square$

Can calculate next line of the triangle by using the fact that $\binom{n}{0} = \binom{n}{n} = 1$ and Pascal's Identity.

$$
\begin{array}{llllllllll}
n = 0: & & & & & \binom{0}{0} & & & & \\
n = 1: & & & & \binom{1}{0} & & \binom{1}{1} & & & \\
n = 2: & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
n = 3: & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
n = 4: & 1 & & \binom{3}{0} + \binom{3}{1} & & \binom{3}{1} + \binom{3}{2} & & \binom{3}{2} + \binom{3}{3} & & 1
\end{array}
$$

Can calculate next line of the triangle by using the fact that $\binom{n}{0} = \binom{n}{n} = 1$ and Pascal's Identity.

$n = 0$:					$\binom{0}{0}$					
$n = 1$:				$\binom{1}{0}$		$\binom{1}{1}$				
$n = 2$:			$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$			
$n = 3$:		$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$		
$n = 4$:	$\binom{4}{0}$		$\binom{4}{1}$		$\binom{4}{2}$		$\binom{4}{3}$		$\binom{4}{4}$	

C. Stirling's formula

Many interesting applications involve the binomial coefficients $\binom{n}{k}$ for large values of m and n. It is therefore sometimes useful to know Stirling's approximation

$$n! \sim n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}$$

for large integers n, meaning that

$$\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}} = 1.$$

As an application, we compute that

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} \sim \frac{(2n)^{2n+\frac{1}{2}} e^{-2n} \sqrt{2\pi}}{(n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi})^2} = \frac{2^{2n}}{\sqrt{\pi n}}$$

for large n.

Section 6

More Counting: Balls into Boxes

Introduction: the 12-Fold Way

Now that we have talked about permutations and combinations, we will discuss a general framework that gives structure to all of these counting problems.

Imagine that you have a collection of boxes and a pile of balls. You want to distribute the balls into the boxes.

We will study 3 different rules for the mapping assigning the balls to the boxes.

- There must be at most 1 ball in each box: the mapping is **injective**.
- There must be at least 1 ball in each box: the mapping is **surjective**
- There are no restrictions on how many balls are in each box; the mapping is **unrestricted**.

("Injective" means one-to-one: "surjective" means onto.)

Now, there are 2 possibilities regarding the collection of balls:

- The balls could be **indistinguishable** from each other (i.e. every ball is exactly the same); or
- the balls could be **distinguishable** from each other (suppose e.g. the balls are painted different colors).

Also, there are 2 possibilities for the boxes:

- The boxes could either be **indistinguishable** from each other, or
- Or, they could **distinguishable** from each other (say, there is a number painted on the side of each box).

Consequently there are $3 \cdot 2 \cdot 2 = 12$ distinct counting problems: we have a 12-fold way!

We will hereafter sometimes use the the term **urn** to refer to the boxes. The advantage is that the words "ball" and "urn" start with different letters.

So let B be a set of balls and let U be a set of boxes (or urns), with

$$|B| = b, \quad |U| = u.$$

Here is a table that we will fill in, showing the number of ways to put the balls into the bins under each circumstance.

Balls	Urns	Arbitrary	Injective (≤ 1)	Surjective (≥ 1)
Dist.	Dist.			
Indist.	Dist.			
Dist.	Indist.			
Indist.	Indist.			

The following 4 sections consider into the various cases as to whether the balls and the urns are distinguishable or not.

A. Distinguishable balls and urns

How many ways are there to put b distinguishable balls in the u distinguishable urns with no restrictions on the mapping?

There are u choices of bins for 1st ball AND u choices for bins for 2nd ball AND ... AND u choices for the bth ball. Using the Product Rule, we can see that the number of ways is

$$u^b.$$

Balls	Urns	Arbitrary	Injective (≤ 1)	Surjective (≥ 1)
Dist.	Dist.	u^b		
Indist.	Dist.			
Dist.	Indist.			
Indist.	Indist.			

Example 6.1

Imagine that there are 10 possible pizza toppings and we need to choose which ones to have on our pizza. How many possible different pizza combinations are there? (Rephrase this in terms of balls and bins!)

The balls are the pizza toppings. Each topping can be on the pizza or not on the pizza, so it goes in one of two "bins". Since $b = 10$ *and* $u = 2$*, there are* 2^{10} *different combinations.* □

Example 6.2

How many strings of six letters are there?

There are 26 letters (bins) and 6 characters in the string (balls). So there are 26^6 *different strings of 6 letters.* □

Example 6.3

How many ways are there to assign three jobs to five employees if each employee can be given more than one job?

There are 5 employees (bins) and 3 jobs (balls). So there are 5^3 *different ways.* □

How many ways are there to put b distinguishable balls in the u distinguishable bins with no more than 1 ball in each bin?

There are u choices of bins for 1st ball AND $u-1$ choices for bins for 2nd ball AND ... AND $u-(b-1)$ choices for the bth ball. Using the Product Rule, we can see that there are

$$(u)_b = u \cdot (u-1) \cdots (u-b+1)$$

ways.

Balls	bins	Arbitrary	Injective (≤ 1)	Surjective (≥ 1)
Dist.	Dist.	u^b	$(u)_b$	
Indist.	Dist.			
Dist.	Indist.			
Indist.	Indist.			

Example 6.4

There are five people and you need to put them in a line for a picture. How many different ways could they line up?

Think of each person as a ball and each spot in the line as an bin. Since $b = 5$ *and* $u = 5$*, there are* $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ *ways.* □

Example 6.5

How many ways can you deal five cards from a deck of 52 (order matters)?

Since $b = 5$ *and* $u = 52$*, there are* $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ *ways.* □

B. Indistinguishable balls, distinguishable urns

How many ways are there to put b indistinguishable balls in the u distinguishable bins with no more than 1 ball in each bin?

When the balls are distinguishable, there are $(u)_b = u \cdot (u-1) \cdots (u-b+1) \cdot (u-b+1)$ ways. Let N be the number of ways in the indistinguishable ball case. Then, by the Product Rule, $(u)_b = b!N$ since there are $b!$ different ways of permuting the b balls. Hence, we can see that there are $N = (u)_b / b! = \frac{u \cdot (u-1) \cdots (u-b+1)}{b!} = \binom{u}{b}$ ways.

Balls	bins	Arbitrary	Injective (≤ 1)	Surjective (≥ 1)
Dist.	Dist.	u^b	$(u)_b$	
Indist.	Dist.		$\binom{u}{b}$	
Dist.	Indist.			
Indist.	Indist.			

Example 6.6

A factory makes auto parts. Sometimes the machines make defective parts. Imagine that in an hour, the factory produces 100 *parts (bins) of which* 5 *are defective (balls). How many different orderings of defective and non-defective parts are there on the assembly line?*

Imagine those 100 parts as spots (bins) that could be defective (have a ball). Then $u = 100$ *and* $b = 5$*. Since the defective parts are indistinguishable, the number of orderings is* $\binom{100}{5}$. □

Example 6.7

How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

The first person can be dealt 5 cards in $\binom{52}{5}$ *ways and the second can be dealt 5 cards in* $\binom{47}{5}$ *ways and the third can be dealt 5 cards in* $\binom{42}{5}$ *ways and finally the fourth can be dealt 5 cards in* $\binom{37}{5}$ *ways.*
In total there are $\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}$ *ways.* $\square$

How many ways are there to put b indistinguishable balls in the u distinguishable bins with no restrictions on the mapping ?

We need to choose b bins with replacement since the b balls are the same. To think about this problem, we will introduce a tool such that there is a bijection to the balls and bins. Imagine that we represent each ball with $\star$. We need divide the balls into specific bins using a vertical line $|$.

For example, let $b = 5$ and $u = 3$. The balls can be represented by

$$\star \quad \star \quad \star \quad \star \quad \star$$

For the particular partition into 3 boxes, we only need 2 vertical lines,

$$\star \quad | \quad | \quad \star \quad \star \quad \star \quad \star$$

In this case, there is 1 ball in the first box, 0 in the second, and 4 in the third box.

To imagine all possible configurations, we can see that there are $u + b - 1$ places to put b stars and the rest will automatically be vertical lines.

$$\underline{\qquad}\quad \underline{\qquad}\quad \underline{\qquad}\quad \underline{\qquad}\quad \underline{\qquad}\quad \underline{\qquad}\quad \underline{\qquad}$$

$$\underline{\ \star\ }\quad \underline{\ \ -\ \ }\quad \underline{\ \ -\ \ }\quad \underline{\ \star\ }\quad \underline{\ \star\ }\quad \underline{\ \star\ }\quad \underline{\ \star\ }$$

Now, we are back to the framework with b indistinguishable balls and $u^* = u + b - 1$ distinguishable bins with no more than 1 ball in each bin.

There are therefore

$$\binom{u^*}{b} = \binom{u + b - 1}{b}$$

ways to divide up the balls into the bins.

Balls	Urns	Arbitrary	Injective (≤ 1)	Surjective (≥ 1)
Dist.	Dist.	u^b	$(u)_b$	
Indist.	Dist.	$\binom{u+b-1}{b}$	$\binom{u}{b}$	
Dist.	Indist.			
Indist.	Indist.			

Example 6.8

When expand out the expression $(a+b+c)^{20}$, how many terms does it contain?

When expanded, all terms can be written as $a^p b^q c^r$ with $p+q+r=20$. The number of terms is the number of solutions of the equation $p+q+r=20$ with p,q,r as positive integer unknowns.
Think of p,q,r as the three bins; we have 20 balls that need to go into the bins. The values of p,q,r will be the number of balls in each of the labeled bins. The number of solutions to $p+q+r=20$ is equivalent to the number of ways you can put 20 indistinguishable balls into 3 distinguishable bins. Thus the number of ways is $\binom{3+20-1}{20}=\binom{22}{20}$. □

Example 6.9

A bagel shop has 8 types of bagels: onion, poppy seed, egg, salty, pumpernickel, sesame seed, raisin, and plain. How many ways are there to choose six bagels? There are 8 types of bagels (bins) and there are

room for 6 in your order (balls) (the order of the bagels doesn't matter). Thus, there are $\binom{8+6-1}{6}$ *ways to choose six bagels.* □

How many ways are there to put b indistinguishable balls in the u distinguishable bins with at least one ball in each bin?

Put one ball in each bin. Now you have $b-u$ balls that can be distributed without restriction.
We already know that there are $\binom{u+b-1}{b}$ ways of putting b balls into u bins without restriction. Let $b^{**}=b-u$, then the number of ways is

$$\binom{u+b^{**}-1}{b^{**}}=\binom{u+(b-u)-1}{b-u}=\binom{b-1}{b-u}.$$

Balls	Urns	Arbitrary	Injective (≤ 1)	Surjective (≥ 1)
Dist.	Dist.	u^b	$(u)_b$	
Indist.	Dist.	$\binom{u+b-1}{b}$	$\binom{u}{b}$	$\binom{b-1}{b-u}$
Dist.	Indist.			
Indist.	Indist.			

Example 6.10

How many terms are contained in $(a+b+c)^{20}$ that contain a, b, and c?

This is similar to the a previous example, except that $p, q, r > 0$. The number of ways is $\binom{20-1}{20-3} = \binom{19}{17}$.

Example 6.11

A bagel shop has 8 types of bagels: onion, poppy seed, egg, salty, pumpernickel, sesame seed, raisin, and plain. How many ways are there to choose a dozen bagels such that you have at least one of each?

There are 8 types of bagels (bins) and there are room for 12 in your order (balls) (the order of the bagels doesn't matter), but you need at least one ball in each bin. Thus, there are $\binom{12-1}{12-8} = \binom{11}{4}$ ways to choose six bagels. □

C. Distinguishable balls, indistinguishable urns

How many ways are there to put b distinguishable balls in the u indistinguishable bins? Since the bins are indistinguishable, this means dividing the balls into unordered subsets.

DEFINITION

The **Stirling numbers** *of the second kind $S(b,u)$ count the number of ways to partition a set of b elements into u nonempty (and nondistinct) subsets.*

THEOREM

The Stirling numbers satisfy the recurrence

$$S(b+1,u) = uS(b,u) + S(b,u-1), \tag{1}$$

with $S(0,0)=1$ and $S(b,0)=S(0,u)=0$.

Proof.

It is easy to see that $S(0,0) = 1, S(b,0) = S(0,u) = 0$.

If we have a set with, say, $b+1$ people, let us designate one particular person.

1. We remove the designed person from the group, and then subdivide the remaining b people into $u-1$ nonempty subsets, in $S(b,u-1)$ ways. We then have the designed person form her own singleton set in addition.
2. Alternatively, we can subdivide the remaining b people into u nonempty subsets, in $S(b,u)$ ways. The designated person can then join any of these u subsets. There are $uS(b,u)$ ways to do this.

Therefore $S(b+1,u) = uS(b,u) + S(b,u-1)$.

□

REMARK: The recurrence relation above can be solved, giving

$$\boxed{S(b,u) = \frac{1}{u!}\sum_{j=0}^{u}(-1)^j \binom{u}{j}(u-j)^n}.$$

Surjective: Partition b balls into u unordered, nonempty subsets.
Number of ways $= S(b, u)$ (Stirling number of the second kind).

Arbitrary: Partition b balls into u unordered subsets.
Number of ways $= \sum_{i=1}^{u} S(b, i)$

Injective: Partition b balls into u unordered singleton subsets.
Number of ways $= 1$ if $b \leq u$ and 0 if $b > u$.

Balls	Urns	Arbitrary	Injective (≤ 1)	Surjective (≥ 1)
Dist.	Dist.	u^b	$(u)_b$	
Indist.	Dist.	$\binom{u+b-1}{b}$	$\binom{u}{b}$	$\binom{b-1}{b-u}$
Dist.	Indist.	$\sum_{i=1}^{u} S(b, i)$	1 if $b \leq u$, 0 if $b > u$	S(b,u)
Indist.	Indist.			

Example 6.12

How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

We could put all four employees into 1 office $\{\{A, B, C, D\}\}$.
We could 3 employees in one office and 1 in another $\{\{A, B, C\}, \{D\}\}$, $\{\{A, B, D\}, \{C\}\}$, $\{\{A, C, D\}, \{B\}\}$, $\{\{B, C, D\}, \{A\}\}$.
We could put 2 employees in 1 office and 2 in another $\{\{A, B\}, \{C, D\}\}$, $\{\{A, C\}, \{B, D\}\}$, $\{\{A, D\}, \{B, C\}\}$.
Finally, we could put two employees in 2 office and the other two in the own office $\{\{A, B\}, \{C\}, \{D\}\}$, $\{\{A, C\}, \{B\}, \{D\}\}$,$\{\{B, C\}, \{A\}, \{D\}\}$, $\{\{A, D\}, \{C\}, \{B\}\}$, $\{\{D, B\}, \{C\}, \{A\}\}$, $\{\{C, D\}, \{A\}, \{B\}\}$.
In total there are 14 ways to put 4 employees into 3 offices $(S(4, 1) + S(4, 2) + S(4, 3))$. □

Example 6.13

How many ways are there to put four different employees into three indistinguishable offices, when each office must contain at least one employee?

The only solution is to put two employees in 2 office and the other two in the own office $\{\{A, B\}, \{C\}, \{D\}\}$, $\{\{A, C\}, \{B\}, \{D\}\}$, $\{\{B, C\}, \{A\}, \{D\}\}$, $\{\{A, D\}, \{C\}, \{B\}\}$, $\{\{D, B\}, \{C\}, \{A\}\}$, $\{\{C, D\}, \{A\}, \{B\}\}$.
In total there are 6 ways to put 4 employees into 3 offices with at least one in each office ($S(4, 3)$). $\square$

Next, assume bins are indistinguishable, then there are $S(b, u)$ ways to partition the b balls. We know that there are $u!$ permutations of the bins. Therefore, there are $u!S(b, u)$ ways to put b balls in u bins with at least one in each.

Balls	Urns	Arbitrary	Injective (≤ 1)	Surjective (≥ 1)
Dist.	Dist.	u^b	$(u)_b$	$u!S(b, u)$
Indist.	Dist.	$\binom{u+b-1}{b}$	$\binom{u}{b}$	$\binom{b-1}{b-u}$
Dist.	Indist.	$\sum_{i=1}^{u} S(b, i)$	1 if $b \leq u$, 0 if $b > u$	S(b,u)
Indist.	Indist.			

D. Indistinguishable balls and urns

How many ways are there to put b indistinguishable balls in the u indistinguishable boxes?
As the balls and bins are indistinguishable, this is equivalent partitioning a natural number b into a sum of u natural numbers.

DEFINITION

The **partition function** $p_u(b)$ *counts the number of partitions of b into u parts (the number of distinct ways to write b as the sum of u positive integers).*
Note that

$$\boxed{p_u(b) = p_{u-1}(b-1) + p_u(b-u)}$$

where $p_1(b) = p_b(b) = 1$ *and* $p_u(b) = 0$ *if* $u > b$.

Surjective: Partition b into a sum of u positive integers.
Number of ways $= p_u(b)$.

Arbitrary: Partition b into a sum of at most u positive integers.
Number of ways $= \sum_{i=1}^{u} p_i(b)$.

Injective: Partition b balls into u unordered singleton subsets.
Number of ways $= 1$ if $b \leq u$ and 0 if $b > u$.

Balls	Urns	Arbitrary	Injective (≤ 1)	Surjective (≥ 1)
Dist.	Dist.	u^b	$(u)_b$	$u!S(b,u)$
Indist.	Dist.	$\binom{u+b-1}{b}$	$\binom{u}{b}$	$\binom{b-1}{b-u}$
Dist.	Indist.	$\sum_{i=1}^{u} S(b,i)$	1 if $b \leq u$, 0 if $b > u$	S(b,u)
Indist.	Indist.	$\sum_{i=1}^{u} p_i(b)$	1 if $b \leq u$, 0 if $b > u$	$p_u(b)$

Example 6.14

How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

We can enumerate all the ways to pack the books:
6
5,1
4,2
4,1,1
3,3
3,2,1
3,1,1,1
2,2,2
2,2,1,1
There are 9 ways to pack them $(p_1(6) + p_2(6) + p_3(6) + p_4(6))$. □

Example 6.15

How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books and there is at least one book in each box?

We can enumerate all the ways to pack the books:
3,1,1,1
2,2,1,1
There are 2 ways to pack them ($p_4(6)$). □

Section 7

Algorithms

A. Algorithms

DEFINITION

An **algorithm** *is a finite sequence of precise instructions for performing a computation or solving a problem, together with a proof or verification that the instructions will produce the correct output.*

Properties of algorithms

- **Input:** information supplied from outside.
- **Output:** information produced by algorithm (depends on input).
- **Definiteness:** each step is defined precisely.
- **Correctness:** each set of input values generates the correct output.
- **Finiteness:** output produced in a finite amount of time.
- **Generality:** applies to all problems of a certain form.

Example 7.1

TASK: *Tie a ribbon on the shortest tree in a row of experimentally treated fir trees.*

ALGORITHM:

1. *Measure the tree at one end of the row and tie the ribbon to it.*
2. *Measure the next tree down. If it is shorter than the tree with the ribbon, move the ribbon to the new tree. Otherwise, leave the ribbon where it is.*
3. *Repeat step 2 until the end of the row is reached.*

Let's see how our simple algorithm satisfies the properties of an algorithm:

- The input is the row of trees.
- The output is the ribbon being tied to the shortest tree.
- Each step is clearly stated.
- It is easy to verify that this algorithm always places the ribbon correctly.
- It is easy to see that the algorithm takes a finite number of steps and thus finishes in finite time.
- The algorithm will work for *any* row of trees.

B. Stable allocations

To illustrate an algorithm for a more substantial problem, we discuss now **stable allocations**, in the context of the following example:

TASK: Suppose that N men and N women are to marry, and that each person has his/her own preference rankings of those of the opposite sex, as to their desirability as a partner.
In other words, each man has his own ordered list of this first choice for a wife, second choice, etc; and each woman likewise has her own ordered list of preferences for a husband.

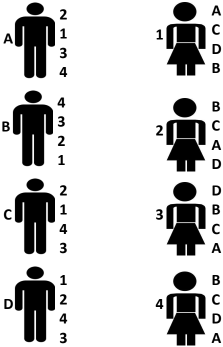
Is it possible to find a assignment of each man to precisely one woman, so that these marriages are all **stable**? "Stable" means that there are no pairs of a man and a woman, who are not married to each other, but both of whom would prefer the other to their own partners.

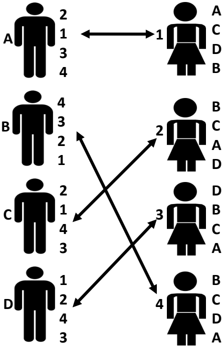
ALGORITHM:[1]

1. Each man makes a proposal to his top ranked woman.
2. Each woman who has received at least one proposal, keeps her top choice waiting and rejects the other proposals (if any)
3. Each man not now kept waiting makes a proposal to the next highest ranked woman on his list.
4. Each woman selects her top choice amongst the new proposals and any previous one kept waiting, and rejects all others.
5. Continue until each woman has precisely one suitor in waiting, whom she now marries.

[1] D. Gale and L. S. Shapley: "College Admissions and the Stability of Marriage", American Math Monthly 69, 1962. Shapley shared the 2012 Nobel Memorial Prize in Economic Sciences for this algorithm.

Example


A 2 1 3 4
B 4 3 2 1
C 2 1 4 3
D 1 2 4 3
1 A C D B
2 B C A D
3 D B C A
4 B C D A


A 2 1 3 4
B 4 3 2 1
C 2 1 4 3
D 1 2 4 3
1 A C D B
2 B C A D
3 D B C A
4 B C D A

The matching algorithm in action

Woman	Round 1	Round 2	Round 3	Round 4	Round 5
1	D	A,D	A	A	A
2	A,C	C	C,D	C	C
3					D
4	B	B	B	B,D	B

- Woman 2 rejects A at the end of Round 1
- Woman 1 rejects D at the end of Round 2
- Woman 2 rejects D at the end of Round 3
- Woman 4 rejects D at the end of Round 4

Notice that the outcome of this process did not depend at all upon the preferences of Woman 3.