Bruhat Interval polytopes

Emmanuel Tsukerman

University of California, Berkeley

July 6, 2015

Based on a joint paper with L. Williams
Outline

1 Preliminaries - Bruhat order

2 Background - Richardson varieties, Moment map

3 Bruhat Interval polytopes
Preliminaries
I will be considering the symmetric group $S_n$ and its presentation as a Coxeter group.
I will be considering the symmetric group $S_n$ and its presentation as a Coxeter group

Generators are:

$$s_i = (i, i + 1), \quad i = 1, 2, \ldots, n - 1$$
I will be considering the symmetric group $S_n$ and its presentation as a Coxeter group.

Generators are:

$$s_i = (i, i + 1), \quad i = 1, 2, \ldots, n - 1$$

Relations are:

- $s_i^2 = e$
- $s_is_j = s_j s_i$ if $j \neq i \pm 1$
- $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$
Given a word $w = s_{i_1} s_{i_2} \ldots s_{i_q}$ in $S_n$, we may apply a sequence of relations to it to obtain a shortest possible word
Given a word $w = s_{i_1} s_{i_2} \ldots s_{i_q}$ in $S_n$, we may apply a sequence of relations to it to obtain a shortest possible word

- Such a shortest expression for $w$ is called a \textit{reduced expression} for $w$
Given a word $w = s_{i_1}s_{i_2} \ldots s_{i_q}$ in $S_n$, we may apply a sequence of relations to it to obtain a shortest possible word

- Such a shortest expression for $w$ is called a *reduced expression* for $w$

Example:

$$s_1s_2s_1s_2 = s_2s_1s_2s_2 = s_2s_1.$$
Bruhat order

Let \( w = s_1 s_2 \ldots s_q \) be a reduced expression. Define

\[
    u \leq w
\]

if there exists a reduced expression

\[
    u = s_{i_1} s_{i_2} \ldots s_{i_k}, \quad 1 \leq i_1 < \ldots < i_k \leq q.
\]
Bruhat order

Let \( w = s_1 s_2 \ldots s_q \) be a reduced expression. Define

\[
u \leq w\]

if there exists a reduced expression

\[
u = s_{i_1} s_{i_2} \ldots s_{i_k}, \quad 1 \leq i_1 < \ldots < i_k \leq q.
\]

Example

- \( v = s_1 s_2 s_1 = 321, \ u = s_2 = 132 \)
Bruhat order

Let $w = s_1 s_2 \ldots s_q$ be a reduced expression. Define

$$u \leq w$$

if there exists a reduced expression

$$u = s_{i_1} s_{i_2} \ldots s_{i_k}, \quad 1 \leq i_1 < \ldots < i_k \leq q.$$  

Example

- $v = s_1 s_2 s_1 = 321$, $u = s_2 = 132$
- words for $u, v$ are reduced
Bruhat order

Let \( w = s_1 s_2 \ldots s_q \) be a reduced expression. Define

\[ u \leq w \]

if there exists a reduced expression

\[ u = s_{i_1} s_{i_2} \ldots s_{i_k}, \quad 1 \leq i_1 < \ldots < i_k \leq q. \]

Example

- \( v = s_1 s_2 s_1 = 321, \ u = s_2 = 132 \)
- words for \( u, v \) are reduced
- \( v = s_1 s_2 s_1, \ u = s_2 \)
Bruhat order

Let $w = s_1 s_2 \ldots s_q$ be a reduced expression. Define

$$u \leq w$$

if there exists a reduced expression

$$u = s_{i_1} s_{i_2} \ldots s_{i_k}, \quad 1 \leq i_1 < \ldots < i_k \leq q.$$ 

Example

- $v = s_1 s_2 s_1 = 321$, $u = s_2 = 132$
- words for $u, v$ are reduced
- $v = s_1 s_2 s_1$, $u = s_2$
- $\implies v \geq u$
Example (Tricky)

- \( v = s_1 s_2 s_1, \ u = s_2 s_1 s_2 \)
Example (Tricky)

- $v = s_1 s_2 s_1$, $u = s_2 s_1 s_2$
- Doesn’t look like $v \geq u$ or $u \geq v$
Example (Tricky)

- $v = s_1 s_2 s_1$, $u = s_2 s_1 s_2$
- Doesn’t look like $v \geq u$ or $u \geq v$
- But actually, $v = s_1 s_2 s_1 = s_2 s_1 s_2 = u$ (braid relation)
Example (Tricky)

- $v = s_1s_2s_1$, $u = s_2s_1s_2$
- Doesn’t look like $v \geq u$ or $u \geq v$
- But actually, $v = s_1s_2s_1 = s_2s_1s_2 = u$ (braid relation)

When $v$ covers $u$, i.e., $v \geq u$ in Bruhat order and there is no $z \neq u$, $v$ such that $u \leq z \leq v$, we will write

$$v \gtrdot u.$$
Quick word about Schubert varieties and Bruhat order
Quick word about Schubert varieties and Bruhat order

For $\nu \in S_{n+1}$, there is an associated Schubert variety $\Omega_\nu$. 
For $v \in S_{n+1}$, there is an associated Schubert variety $\Omega_v$.

It is well-known that

$$u \leq v \iff \Omega_v \subset \Omega_u$$
Bruhat order on $S_4$

**Figure:** Hasse Diagram for $S_4$ in Bruhat order
Background
Bruhat Interval polytope (BIP)

For $u \leq v$, define

$$P_{u,v} := \text{Conv}(\{z \in S_n : u \leq z \leq v\})$$
Bruhat Interval polytope (BIP)

For \( u \leq v \), define

\[
P_{u,v} := \text{Conv}\{z \in S_n : u \leq z \leq v\}\]

Here we are viewing an element \( z = (z(1), z(2), \ldots, z(n)) \) of \( S_n \) as a vector in \( \mathbb{R}^n \)
Bruhat Interval polytope (BIP)

For $u \leq v$, define

$$P_{u,v} := \text{Conv}\{z \in S_n : u \leq z \leq v\}$$

Here we are viewing an element $z = (z(1), z(2), \ldots, z(n))$ of $S_n$ as a vector in $\mathbb{R}^n$
Permutahedron is a BIP

When \( u = (1, 2, \ldots, n) \) and \( v = (n, n - 1, n - 2, \ldots, 1) \), \( P_{u,v} \) is the permutahedron!
Permutahedron is a BIP

When $u = (1, 2, \ldots, n)$ and $v = (n, n - 1, n - 2, \ldots, 1)$, $P_{u,v}$ is the permutahedron!

- The permutahedron is the convex hull of all permutations of $(1, 2, \ldots, n)$
Permutahedron is a BIP

When $u = (1, 2, \ldots, n)$ and $v = (n, n-1, n-2, \ldots, 1)$, $P_{u,v}$ is the permutahedron!

- The permutahedron is the convex hull of all permutations of $(1, 2, \ldots, n)$
Motivation

Motivation comes from studying an integrable system called “The Full Kostant-Toda Hierarchy” [Kodama & Williams, 2013]
Motivation comes from studying an integrable system called “The Full Kostant-Toda Hierarchy” [Kodama & Williams, 2013]

- The positive parts $\mathcal{R}_{u,v}^{>0}$ of Richardson varieties stratify the totally nonnegative part of the flag variety
Motivation

Motivation comes from studying an integrable system called “The Full Kostant-Toda Hierarchy” [Kodama & Williams, 2013]

- The positive parts $\mathcal{R}_{u,v}^{>0}$ of Richardson varieties stratify the totally nonnegative part of the flag variety
- The moment map $\mu$ (a notion from symplectic geometry) maps $\mathcal{R}_{u,v}^{>0}$ to a polytope
Motivation

Motivation comes from studying an integrable system called “The Full Kostant-Toda Hierarchy” [Kodama & Williams, 2013]

- The positive parts $\mathcal{R}_{u,v}^>^0$ of Richardson varieties stratify the totally nonnegative part of the flag variety
- The moment map $\mu$ (a notion from symplectic geometry) maps $\mathcal{R}_{u,v}^>^0$ to a polytope
- K+W show this polytope is a Bruhat Interval polytope:

$$\mu(\mathcal{R}_{u,v}) = \mu(\mathcal{R}_{u,v}^>^0) = P_{u,v}$$
Properties of BIPs

Bruhat Interval polytopes are:
Properties of BIPs

Bruhat Interval polytopes are:

- Minkowski sums of matroid (positroid) polytopes
Properties of BIPs

Bruhat Interval polytopes are:

- Minkowski sums of matroid (positroid) polytopes
- Generalized permutohedra
Properties of BIPs

Bruhat Interval polytopes are:

- Minkowski sums of matroid (positroid) polytopes
- Generalized permutohedra
- flag matroid polytopes
Bruhat Interval polytopes
Theorem (T-Williams)

A face of a Bruhat Interval polytope is a Bruhat Interval polytope
Theorem (T-Williams)

A face of a Bruhat Interval polytope is a Bruhat Interval polytope

Example:
For comparison, a face of a permutahedron is not necessarily a permutahedron
For comparison, a face of a permutahedron is not necessarily a permutahedron - it is, however, always a product of permutahedra.
Ideas used in the proof

A Generalized lifting property

It is helpful to place this property in context

Define

$D_R(x) := \{ s \in S : xs \succ x \}$

("right descents")

In $S_n$, $D_R(x^{(1)}x^{(2)}...x^{(n)}) = \{ s_i : x^{(i)} > x^{(i+1)} \}$

For example, $D_R(1324) = \{ s_2 \}$

The following classical property characterizes Bruhat order:

Proposition - lifting property

Suppose $v > u$ and $s \in D_R(v) \setminus D_R(u)$. Then $v \geq us$ and $vs \geq u$.

Figure: Lifting property
A Generalized lifting property

Define \( D_R(x) := \{ s \in S : x_s \preceq x \} \) ("right descents") in \( S_n \), where \( D_R(x^{(1)}x^{(2)}...x^{(n)}) = \{ s_i : x^{(i)} > x^{(i+1)} \} \). For example, \( D_R(1324) = \{ s_2 \} \).

The following classical property characterizes Bruhat order:

**Proposition - lifting property**

Suppose \( v > u \) and \( s \in D_R(v) \setminus D_R(u) \). Then \( v \geq us \) and \( vs \geq u \).
A Generalized lifting property
It is helpful to place this property in context

Define
$$D_R(x) := \{ s \in S : xs \succ x \}$$
("right descents")

In $S_n$, $D_R(x_1 x_2 \ldots x_n) = \{ s_i : x_i > x_{i+1} \}$

For example, $D_R(1324) = \{ s_2 \}$

The following classical property characterizes Bruhat order:

Proposition - lifting property
Suppose $v > u$ and $s \in D_R(v) \setminus D_R(u)$. Then $v \geq us$ and $vs \geq u$.

Figure: Lifting property
Ideas used in the proof

A Generalized lifting property
It is helpful to place this property in context

- Define $D_R(x) := \{ s \in S : xs \preceq x \}$ ("right descents")
**A Generalized lifting property**

It is helpful to place this property in context

- Define $D_R(x) := \{s \in S : xs \triangleleft x\}$ ("right descents")
- In $S_n$, $D_R(x(1)x(2)\ldots x(n)) = \{s_i : x(i) > x(i + 1)\}$
A Generalized lifting property

It is helpful to place this property in context

- Define $D_R(x) := \{ s \in S : xs \preceq x \}$ (“right descents”)
- In $S_n$, $D_R(x(1)x(2)\ldots x(n)) = \{ s_i : x(i) > x(i + 1) \}$
- For example, $D_R(1324) = \{ s_2 \}$
A Generalized lifting property

It is helpful to place this property in context

- Define $D_R(x) := \{s \in S : xs \lessdot x\}$ (“right descents”)
- In $S_n$, $D_R(x(1)x(2)\ldots x(n)) = \{s_i : x(i) > x(i + 1)\}$
- For example, $D_R(1324) = \{s_2\}$

The following classical property characterizes Bruhat order:
Ideas used in the proof

**A Generalized lifting property**

It is helpful to place this property in context

- Define $D_R(x) := \{ s \in S : xs \prec x \}$ (“right descents”)
- In $S_n$, $D_R(x(1)x(2)\ldots x(n)) = \{ s_i : x(i) > x(i + 1) \}$
- For example, $D_R(1324) = \{s_2\}$

The following classical property characterizes Bruhat order:

**Proposition - lifting property**

Suppose $v > u$ and $s \in D_R(v) \setminus D_R(u)$. Then $v \geq us$ and $vs \geq u$. 
The lifting property is also a means to understanding $R$-polynomials (and consequently, Kazhdan-Lustzig polynomials)
The lifting property is also a means to understanding $R$-polynomials and consequently, Kazhdan-Lustzig polynomials. KL-polynomials appear in: Canonical bases, Verma modules, rep. theory of semisimple alg. groups, intersection cohomology,...
The lifting property is also a means to understanding $R$-polynomials and consequently, Kazhdan-Lustzig polynomials. KL-polynomials appear in: Canonical bases, Verma modules, rep. theory of semisimple alg. groups, intersection cohomology,

**Theorem-Definition [Kazhdan & Lusztig, 1979]**

There is a unique family of polynomials \( \{ R_{u,v}(q) \}_{u,v \in S_n} \subset \mathbb{Z}[q] \) satisfying the following conditions:

(i) \( R_{u,v} = 0 \), if \( u \not\leq v \).

(ii) \( R_{u,v} = 1 \), if \( u = v \).

(iii) If \( s \in D_R(v) \), then

\[
R_{u,v}(q) = \begin{cases} 
R_{us,vs}(q) & \text{if } s \in D_R(u), \\
nR_{us,vs}(q) + (q - 1)R_{u,vs}(q) & \text{if } s \not\in D_R(u).
\end{cases}
\]
Unfortunately, when $D_R(v) \setminus D_R(u) = \emptyset$, the lifting property doesn’t provide information.
Unfortunately, when \( D_R(v) \setminus D_R(u) = \emptyset \), the lifting property doesn’t provide information.

Fortunately,
**Theorem - Generalized lifting property (T-Williams)**

Suppose $u, v \in S_n$ with $v > u$.

1. Any transposition $t = (ij)$ with the interval $[i, j]$ being minimal with respect to the property
   \[
   v_i > v_j, \quad u_i < u_j
   \]
   satisfies
   \[
   v > vt, \quad ut > u, \quad v \geq ut \quad vt \geq u.
   \]

2. Such a transposition $t = (ij)$ always exists.

---

**Figure:** Generalized lifting property
Example:

\[ v = 3241 \]

\[ t = (24) \quad (12) \]

\[ 3142 \quad 2341 \]

\[ (14) \quad t = (24) \]

\[ u = 2143 \]
By understanding Generalized lifting, one can show

**Theorem (T-Williams)**

Given $u, v \in S_n$ with $v \geq u$, there exists a transposition $t$ such that

$$R_{u, v}(q) = qR_{ut, vt}(q) + (q - 1)R_{u, vt}(q)$$
By understanding Generalized lifting, one can show

**Theorem (T-Williams)**

Given $u, v \in S_n$ with $v \geq u$, there exists a transposition $t$ such that

$$v^t u, v^t u = u$$

and

$$R_u, v(q) = qR_{u^t, v^t}(q) + (q-1)R_{u, v^t}(q)$$

Recall, we used

$$R_u, v(q) = qR_{u^s, v^s}(q) + (q-1)R_{u, v^s}(q)$$

to define the $R$-polynomials.
Ideas used in the proof

Next idea used in the proof:

Theorem ([Björner & Wachs, 1982])

Let $u, v \in W$. The order complex of $(u, v)$ is PL homeomorphic to $S^\ell(u, v) - 2$. 

Example with $\ell(u, v) = 2$
Ideas used in the proof

Next idea used in the proof: Topology of Bruhat order
Ideas used in the proof

Next idea used in the proof: Topology of Bruhat order

**Theorem ([Björner & Wachs, 1982])**

Let \( u, v \in W \). The order complex of \((u, v)\) is PL homeomorphic to \( S^{\ell(u,v) - 2} \).
Ideas used in the proof

Next idea used in the proof: Topology of Bruhat order

**Theorem ([Björner & Wachs, 1982])**

Let $u, v \in W$. The order complex of $(u, v)$ is PL homeomorphic to $S^{\ell(u, v) - 2}$.

Example with $\ell(u, v) = 2$
Further results - dimension of a Bruhat Interval polytope

Say we have a collection $C = \{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\}$ of transpositions in $S_n$. We can assign to such a collection a partition $B$ of $[n]$ as follows: $a_i$ and $b_i$ are in the same block $B$ is the finest such partition.

Example $n = 6$, $C = \{1, 2\}, \{2, 3\}, \{4, 6\}$

$B = 123|46|5$
Further results - dimension of a Bruhat Interval polytope

Say we have a collection $C = \{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\}$ of transpositions in $S_n$
Say we have a collection \( C = \{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\} \) of transpositions in \( S_n \)

We can assign to such a collection a partition \( B \) of \([n] \) as follows:
Say we have a collection $C = \{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\}$ of transpositions in $S_n$.

We can assign to such a collection a partition $B$ of $[n]$ as follows:

- $a_i$ and $b_i$ are in the same block.
Further results - dimension of a Bruhat Interval polytope

Say we have a collection \( C = \{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\} \) of transpositions in \( S_n \).

We can assign to such a collection a partition \( B \) of \([n]\) as follows:

- \( a_i \) and \( b_i \) are in the same block
- \( B \) is the finest such partition
Further results - dimension of a Bruhat Interval polytope

Say we have a collection $C = \{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\}$ of transpositions in $S_n$

We can assign to such a collection a partition $B$ of $[n]$ as follows:

- $a_i$ and $b_i$ are in the same block
- $B$ is the finest such partition

**Example**

- $n = 6$, $C = \{(1, 2), (2, 3), (4, 6)\}$
Further results - dimension of a Bruhat Interval polytope

Say we have a collection $C = \{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\}$ of transpositions in $S_n$

We can assign to such a collection a partition $B$ of $[n]$ as follows:

- $a_i$ and $b_i$ are in the same block
- $B$ is the finest such partition

Example

- $n = 6$, $C = \{(1, 2), (2, 3), (4, 6)\}$
- $B = 123|46|5$
Consider now an interval $I = [u, v]$
Consider now an interval $I = [u, v]$

There are several natural collections of transpositions associated to $I$
$C_{\text{all}}$: all transpositions $t$ s.t. $x \preceq xt$ with $x, xt \in I$
1. $C_{\text{all}}$: all transpositions $t$ s.t. $x \preceq xt$ with $x, xt \in I$

2. $C_{\text{chain}}$: transpositions $t$ coming from a maximal chain starting at $u$ and ending at $v$
1. \( C_{\text{all}} \): all transpositions \( t \) s.t. 
\( x \preceq xt \) with \( x, xt \in I \)

2. \( C_{\text{chain}} \): transpositions \( t \) coming from a maximal chain starting at \( u \) and ending at \( v \)

3. \( C_{\text{atoms}} \): transpositions \( t \) coming from the atoms
1. \( C_{\text{all}} \): all transpositions \( t \) s.t. 
\[ x \triangleleft xt \text{ with } x, xt \in I \]

2. \( C_{\text{chain}} \): transpositions \( t \) coming from a maximal chain starting at \( u \) and ending at \( v \)

3. \( C_{\text{atoms}} \): transpositions \( t \) coming from the atoms

4. \( C_{\text{coatoms}} \): transpositions \( t \) coming from the coatoms
To each of the collections in the previous slide, we can assign a partition:

- $B_{\text{all}}$
- $B_{\text{chain}}$
- $B_{\text{atoms}}$
- $B_{\text{coatoms}}$
Further results

Theorem (T-Williams)

Let $I = [u, v] \subset S_n$. Then

$$B_{u,v} := B_{\text{all}} = B_{\text{chain}} = B_{\text{atoms}} = B_{\text{coatoms}}$$
Further results

**Theorem (T-Williams)**

Let \( I = [u, v] \subseteq S_n \). Then

\[
B_{u,v} := B_{all} = B_{chain} = B_{atoms} = B_{coatoms}
\]

and

\[
dim P_{u,v} = n - |B_{u,v}|.
\]
Further results

Theorem (T-Williams)

Let \( I = [u, v] \subset S_n \). Then

\[
B_{u,v} := B_{\text{all}} = B_{\text{chain}} = B_{\text{atoms}} = B_{\text{coatoms}}
\]

and

\[
\dim P_{u,v} = n - |B_{u,v}|.
\]

Example:
Corollary

The Richardson variety $\mathcal{R}_{u,v}$ in $\text{Fl}_n$ is a toric variety if and only if the number of blocks $\# B_{u,v}$ of the partition $B_{u,v}$ satisfies

$$|B_{u,v}| = n - \ell(v) + \ell(u).$$
Summary
Summary

Bruhat Interval polytopes:
Summary

Bruhat Interval polytopes:

- arise from studying Richardson varieties
Summary

Bruhat Interval polytopes:

- arise from studying Richardson varieties
- generalize the permutahedron
Summary

Bruhat Interval polytopes:

- arise from studying Richardson varieties
- generalize the permutahedron
- faces are also Bruhat Interval polytopes
Summary

Bruhat Interval polytopes:

- arise from studying Richardson varieties
- generalize the permutahedron
- faces are also Bruhat Interval polytopes
- dimension can be computed in terms of partitions and transpositions of Bruhat cover relations
Summary

Bruhat Interval polytopes:
- arise from studying Richardson varieties
- generalize the permutahedron
- faces are also Bruhat Interval polytopes
- dimension can be computed in terms of partitions and transpositions of Bruhat cover relations

The Generalized lifting property is a useful tool for studying Bruhat order and $R$-polynomials
Thank you

Questions?
Theorem (T-Williams)

The diameter of \( P_{u,v} \) is equal to \( \ell(v) - \ell(u) \).
Theorem (T-Williams)

Let \([x, y] \subset [u, v]\). We define the graph \(G_{x,y}^{u,v}\) as follows:

1. The nodes of \(G_{x,y}^{u,v}\) are \(\{1, 2, \ldots, n\}\), with nodes \(i\) and \(j\) identified if they are in the same part of \(B_{x,y}\).
2. There is a directed edge from \(i\) to \(j\) for every \((i, j) \in \bar{T}(y, [u, v])\).
3. There is a directed edge from \(j\) to \(i\) for every \((i, j) \in \bar{T}(x, [u, v])\).

Then the Bruhat Interval polytope \(P_{x,y}\) is a face of the Bruhat Interval polytope \(P_{u,v}\) if and only if the graph \(G_{x,y}^{u,v}\) is a directed acyclic graph.
References

Y. Kodama and L. Williams, The Full Kostant-Toda Hierarchy on the Positive Flag Variety

A. Björner and M. Wachs, Bruhat order of Coxeter groups and shellability. Adv. in Math., 43(1):87-100