Discussion for midterm 2 Worksheet Answers

Some past exam problems

Date: 11/3/2021

MATH 53 Multivariable Calculus

1. Find the maximum and minimum values of the function $(x+1)^2+y^2$ on the ellipse $x^2+y^2/4 = 1$, and say at what points these values occur.

2. Find the area of the region enclosed by the curve $x^2+xy+y^2 = 1$. (Hint: use the substitution $x = u + v\sqrt{3}$, $y = u - v\sqrt{3}$.)

3. Compute $\int_C \vec{F} \cdot d\vec{r}$, where C is the unit circle oriented counterclockwise, and $\vec{F}$ is the vector field
$$\vec{F} = (-y^3 + \sin(\sin x), x^3 + \sin(\sin y)).$$

4. Compute
$$\int_{-2}^{2} \int_{y^2}^{4} y \sin(x^2) dx dy$$

5. Calculate the volume of the region consisting of all points that are inside the sphere $x^2+y^2+z^2 = 4$, below the cone $z = \sqrt{x^2+y^2}$, and above the cone $z = -\sqrt{x^2+y^2}$.

Solution:

1. Let $f(x,y) = (x+1)^2 + y^2$ and $g(x,y) = x^2 + y^2/4$. Then our Lagrange multiplier equations $\nabla f = \lambda \nabla g$, together with our constraint equation, become

$$\begin{aligned} 2(x+1) &= 2\lambda x \\ 2y &= \frac{\lambda}{2} y \\ x^2 + \frac{y^2}{4} &= 1. \end{aligned}$$

   So either $y = 0$ or $\lambda = 4$. If $y = 0$, then $x = \pm 1$, and we have $f(1,0) = 4$ and $f(-1,0) = 0$. If $\lambda = 4$, then we get $2(x+1) = 8x$, so $x = 1/3$. Then $y = \pm\sqrt{32/9} = \pm 4\sqrt{2}/3$. We have $f(1/3, \pm 4\sqrt{2}/3) = 32/9$. So the maximum of f is $32/9$, attained at $(1/3, \pm 4\sqrt{2}/3)$, and the minimum of f is 0, attained at $(-1,0)$.

2. Applying this substitution gives the region enclosed by $3u^2 + 3v^2 = 1$, or equivalently $u^2 + v^2 = 1/3$. The Jacobian of this transformation is $2\sqrt{3}$. So our area is

$$\iint_{x^2+xy+y^2 \leq 1} dxdy = \iint_{u^2+v^2 \leq 1/3} 2\sqrt{3}dudv = 2\sqrt{3} \cdot \pi \frac{1}{3} = \frac{2}{3}\pi\sqrt{3}.$$

3. Write $\vec{F} = (P,Q)$, and let R be the inside of the circle. We have $P_y = -3y^2$ and $Q_x = 3x^2$, so by Green's theorem, we have

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_D 3x^2 + 3y^2 dxdy \\ &= 3\int_0^{2\pi} \int_0^1 r^2 \cdot r dr d\theta \\ &= 3\int_0^{2\pi} \frac{1}{4} d\theta = 3 \cdot \frac{1}{4} \cdot 2\pi = \frac{3}{2}\pi. \end{aligned}$$

4. This is easiest to solve by rewriting the region as $0 \leq x \leq 4$, $-\sqrt{x} \leq y \leq \sqrt{x}$. The integral is then

$$\int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} y\sin(x^2) dydx = \int_0^4 \frac{1}{2}((\sqrt{x})^2 - (\sqrt{x})^2)\sin(x^2) dx = \int_0^4 0 dx = 0.$$

5. By sketching this out, you can see that it's the region given in polar coordinates by $0 \leq \rho \leq 2$, $0 \leq \theta \leq 2\pi$, and $\pi/4 \leq \phi \leq 3\pi/4$. So the volume is

$$\begin{aligned} \int_0^2 \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \rho^2 \sin\phi d\phi d\theta d\rho &= \int_0^2 \rho^2 d\rho \cdot \int_0^{2\pi} d\theta \cdot \int_{\pi/4}^{3\pi/4} \sin\phi d\phi \\ &= \left(\frac{1}{3}2^3 - \frac{1}{3}0^3\right) \cdot 2\pi \cdot \left(-\cos\frac{3\pi}{4} + \cos\frac{\pi}{4}\right) \\ &= \frac{8}{3} \cdot 2\pi \cdot \sqrt{2} = \frac{16}{3}\pi\sqrt{2}. \end{aligned}$$

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.