Discussion 9 Worksheet Answers

Tangent planes (revisited) and optimization

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MATH 53 Multivariable Calculus

1 Tangent Plane

Find the equation of the tangent plane.

(a) $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$ at $(3, 3, 5)$;

> **Solution:** Let $F(x, y, z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2$ then $\nabla F(x, y, z) = \langle 4(x-2), 2(y-1), 2(z-3) \rangle$ so $\nabla F(3, 3, 5) = \langle 4, 4, 4 \rangle$, Hence, the tangent plane is $4(x-3) + 4(y-3) + 4(z-5) = 0$.

(b) $xy^2z^3 = 8$ at $(2, 2, 1)$;

> **Solution:** Let $F(x, y, z) = xy^2z^3$. Then $\nabla F(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$ so $\nabla F(2, 2, 1) = \langle 4, 8, 24 \rangle$. Hence, the tangent plane is $4(x-2) + 8(y-2) + 24(z-1) = 0$.

(c) $x + y + z = e^{xyz}$ at $(0, 0, 1)$.

> **Solution:** Let $F(x, y, z) = x + y + z - e^{xyz}$. Then $\nabla F(x, y, z) = \langle 1 - yze^{xyz}, 1 - xze^{xyz}, 1 - xye^{xyz} \rangle$ so $\nabla F(0, 0, 1) = \langle 1, 1, 1 \rangle$. Hence, the tangent plane is $(x-0) + (y-0) + (z-1) = 0$.

(d) Show that the equation of the tangenet plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1.$$

> **Solution:** $\nabla F(x_0, y_0, z_0) = \langle 2x_0/a^2, 2y_0/b^2, 2z_0/c^2 \rangle$. Then the tangent plane is
>
> $$\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0) = 0.$$
>
> Rearranging, we obtain
>
> $$\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y + \frac{2z_0}{c^2}z = 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right) = 2.$$
>
> Dividing by 2 gives the desired result.

(e) Show that the sum of the $x-$, $y-$, and $z-$intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

Solution: Let (x_0, y_0, z_0) be a point on the surface. The equation of the tangent plane is

$$\frac{1}{2\sqrt{x_0}}(x - x_0) + \frac{1}{2\sqrt{y_0}}(y - y_0) + \frac{1}{2\sqrt{z_0}}(z - z_0) = 0.$$

Rearranging, we obtain

$$\frac{x}{2\sqrt{x_0}} + \frac{y}{2\sqrt{y_0}} + \frac{z}{2\sqrt{z_0}} = \frac{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}{2} = \frac{\sqrt{c}}{2}.$$

The intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$, and $\sqrt{cz_0}$. The sum of the intercepts is $\sqrt{cx_0} + \sqrt{cy_0} + \sqrt{cz_0} = c$.

2 Maxima and Minima

Find the local maximum and minimum values and saddle point(s) of the function.

(a) $f(x, y) = x^2 + y^4 + 2xy$

Solution: We have $f_x = 2x + 2y, f_y = 4y^3 + 2x, f_{xx} = f_{xy} = 2, f_{yy} = 12y^2$. Then $f_x = 0$ implies $y = -x$ and substituting into f_y yields $4y^3 - 2y = 0$. Either $y = 0$ or $y = \pm 1/\sqrt{2}$ so the critical points are $(0, 0), (1/\sqrt{2}, -1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2})$. Now $D(x, y) = 2(12y^2) - 2^2 = 24y^2 - 4$.
$D(0, 0) = -4 < 0$ so $(0, 0)$ is a saddle point. $D(1/\sqrt{2}, -1/\sqrt{2}) = D(-1/\sqrt{2}, 1/\sqrt{2}) = 12 - 4 = 8 > 0$ and $f_{xx} = 2 > 0$ so both points correspond to a local minima.

(b) $f(x, y) = xy + e^{-xy}$

Solution: We have $f_x = y - ye^{-xy}, f_y = x - xe^{-xy}, f_{xx} = y^2e^{-xy}, f_{xy} = 1 + (xy - 1)e^{-xy}, f_{yy} = x^2e^{-xy}$. Then $f_x = 0$ implies $y(1 - e^{-xy}) = 0$ so either $y = 0$ or $x = 0$. If $x = 0$, then $f_y = 0$ for all y so all points of the form $(0, y_0)$ are critical points. If $y = 0$, $f_y = 0$ for all x values so any point of the form $(x_0, 0)$ is a critical point. We have $D(x_0, 0) = 0 = D(0, y_0)$ so the Second Derivative Test gives us no information.
If we let $t = xy$ then $f(x, y) = g(t) = t + e^{-t}$. Then $g'(t) = 1 - e^{-t}$. Then $g'(t) = 0$ only for $t = 0$ and $g''(0) = 1 > 0$ so $g(0) = 1$ is a local minimum. It is an absolute minimum because $g'(t) < 0$ for $t < 0$ and $g'(t) > 0$ for $t > 0$. Thus, $f(x, y) = xy + e^{-xy} \geq 1$ for all (x, y) with equality iff $x = 0$ or $y = 0$. Hence, all the critical points we found correspond to local (and absolute) minima.

3 Challenge

Suppose that the direction derivatives of $f(x, y)$ are known at a given point in two nonparallel directions given by unit vectors $\vec{u}$ and $\vec{v}$. Is it possible to find ∇f at this point? If so, how would you do it?

Solution: Let $\vec{u} = \langle a, b\rangle$ and $\vec{v} = \langle c, d\rangle$. Then $D_{\vec{u}}f = \nabla f \circ \vec{u} = af_x + bf_y$. Similarly $D_{\vec{v}}f = cf_x + df_y$. Since $\vec{u}$ and $\vec{v}$ are not parallel, we can solve this system of linear equations in the two unknowns f_x and f_y. In fact, $\nabla f = \frac{1}{ad-bc}\langle dD_{\vec{u}}f - bD_{\vec{v}}f, aD_{\vec{v}}f - cD_{\vec{u}}f\rangle$.

4 True/False

(a) T F A point that makes $\nabla f = \vec{0}$ corresponds to a critical point.

Solution: **TRUE.** A critical point is obtained when $f_x = 0$ and $f_y = 0$ so $\nabla f = \langle f_x, f_y \rangle = \langle 0, 0 \rangle$.

(b) T F If the second derivative test fails, it is impossible to say anything about the critical point in regards to it being a maxima or minima.

Solution: **FALSE.** See 3(b).

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.