# Discussion 8 Worksheet Answers

## Gradients and directional derivatives

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MATH 53 Multivariable Calculus

## 1 Gradients

Compute the gradients of the following functions.

(a) $f(\theta, \phi) = \cos\theta\cos\phi$

**Solution:**

$$\nabla f = (-\sin\theta\cos\phi, -\cos\theta\sin\phi)$$

(b) $f(x, y) = \arctan(y/x)$

**Solution:**

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2(1+y^2/x^2)}$$
$$\frac{\partial f}{\partial y} = \frac{1}{x(1+y^2/x^2)}$$
$$\nabla f = \frac{1}{x^2+y^2}(-y, x)$$

(c) $f(t, x, y) = \frac{1}{\sqrt{4\pi t}}\exp(-(x-y)^2/4t)$. (Express everything as multiples of $f(t, x, y)$.)

**Solution:**

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\frac{1}{2\sqrt{4\pi}\,t^{3/2}}\exp\left(-\frac{(x-y)^2}{4t}\right) + \frac{1}{\sqrt{4\pi t}}\frac{(x-y)^2}{4t^2}\exp\left(-\frac{(x-y)^2}{4t}\right)\\ &= f(t,x,y)\left(\frac{(x-y)^2-2t}{4t^2}\right)\\ \frac{\partial f}{\partial x} &= \frac{y-x}{2t}f(t,x,y)\\ \frac{\partial f}{\partial y} &= \frac{x-y}{2t}f(t,x,y)\\ \nabla f &= \frac{f(t,x,y)}{4t^2}\left(\left((x-y)^2-2t\right), -2t(x-y), 2t(x-y)\right)\end{aligned}$$

## 2 Directional Derivatives

1. Consider $f(x, y) = e^{-r^4}$ where $r = \sqrt{x^2+y^2}$. Compute its directional derivative at $(0, 0)$ w.r.t. the unit vectors in (polar) directions $\theta = 0, \pi/4, \pi/2$. What about any other angle?

**Solution:** The function is symmetric with respect to rotations around the origin, so all the directional derivatives will be equal. Therefore it is sufficient to compute the derivative for $\theta = 0$, i.e. $\partial/\partial x$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} e^{-(x^2+y^2)^2} = -2(x^2 + y^2) \cdot 2xe^{-(x^2+y^2)^2}$$

Plugging in $x = y = 0$ we see that this is zero.

## 3 Rate of Change

Find the maximum rate of change of $f$ at the given point and direction in which it occurs.

(a) $f(x, y) = 4y\sqrt{x}$ and $(4, 1)$;

**Solution:** $\nabla f(x, y) = \langle 2y/\sqrt{x}, 4\sqrt{x}\rangle$. Then $\nabla f(4, 1) = \langle 1, 8\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(4, 1)| = \sqrt{65}$.

(b) $f(x, y) = \sin(xy)$ at $(1, 0)$;

**Solution:** $\nabla f(x, y) = \langle y\cos(xy), x\cos(xy)\rangle$ so $\nabla f(1, 0) = \langle 0, 1\rangle$. The maximum rate of change is $|\nabla f(1, 0)| = 1$ in the direction $\langle 0, 1\rangle$.

(c) $f(x, y, z) = x/(y + z)$ at $(8, 1, 3)$.

**Solution:** $\nabla f(x, y, z) = \langle 1/(y + z), -x/(y + z)^2, -x/(y + z)^2\rangle$ so $\nabla f(8, 1, 3) = \langle 1/4, -1/2, -1/2\rangle$.

(d) Find all points at which the direction of fastest change of the function $f(x, y) = x^2+y^2-2x-4y$ is $\langle 1, 1\rangle$.

**Solution:** The direction of fastest change is $\nabla f(x, y) = \langle 2x - 2, 2y - 4\rangle$ so we need to find all points $(x, y)$ such that $\nabla f(x, y) = k\langle 1, 1\rangle$. Then $2x - 2 = k$ and $2y - 4 = k$ so $2x - 2 = 2y - 4 \Rightarrow y = x + 1$. Hence, the direction of fastest change is $\langle 1, 1\rangle$ at all points on the line $y = x + 1$.

## 4 True/False

(a) T F Partial derivatives are just a special case of directional derivatives.

**Solution:** **TRUE.** $\partial f/\partial x$ is just the directional derivative in direction $(1, 0)$ and $\partial f/\partial y$ is the derivative in direction $(0, 1)$.

(b) T F Consider $f(x, y) = (\cos x + (1 + x)\tan y)e^{x^2-1+y}$. Then $\frac{d}{dt}f(t^2, t^3) = 0$ at $t = 0$.

**Solution:** **TRUE.** Writing $x(t) = t^2$ and $y(t) = t^3$ we see that $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ at $t = 0$. Now we see from the chain rule that $\frac{d}{dt}f(x(t), y(t)) = 0$ at $t = 0$.

(c) T F Suppose you compute the gradient of $v(s, T) = s/T$, where $s$ has units of miles and $T$ has units of hours. Then the magnitude of $\nabla v$ is well-defined.

**Solution:** **FALSE.** The first component of $\nabla v$ is $\partial v/\partial s$ and has units of inverse hours and the second component is $\partial v/\partial T$, which has units of miles per hour squared. Hence $(\partial v/\partial s)^2$ and $(\partial v/\sigma T)^2$ have different units and cannot be added.

(d) T F When $\vec{u} \perp \nabla f$ at a point, then $D_{\vec{u}} f = 0$ at that point.

**Solution:** **TRUE.** This is because $D_{\vec{u}} f = \nabla f \circ \vec{u} = 0$.

(e) T F If $\vec{u} = \langle a, b \rangle$ is a unit vector and $f$ has continuous second partials, then $D_{\vec{u}}^2 f(x, y) = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$ where $D_{\vec{u}}^2 f = D_{\vec{u}}[D_{\vec{u}} f(x, y)]$.

**Solution:** **TRUE.** We have that $D_{\vec{u}}^2 f = D_{\vec{u}}[D_{\vec{u}} f(x, y)] = D_{\vec{u}}[af_x + bf_y]$. This simplifies to

$$\langle af_{xx} + bf_{xy}, af_{xy} + bf_{yy} \rangle \circ \langle a, b \rangle = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2.$$

Note we needed $f$ to have continous second partials to invoke Clairaut's Theorem.

**Note:** These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.