Discussion 7 Worksheet Answers

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MATH 53 Multivariable Calculus

1 Chain Rule

(a) Compute $\frac{d}{dt}f(x(t),y(t))$ for $f(x,y)=xy^3-x^2y$, $x(t)=t^2+1$, $y(t)=t^2-1$. You don't need to express the final result in terms of t.

Solution:

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=(y^3-2xy)\cdot 2t+(3xy^2-x^2)\cdot 2t=2t(y^3-2xy+3xy^2-x^2)$$

(b) Compute $\frac{d}{dt}f(x(t),y(t))$ for $f(x,y)=\sin x\cos y$, $x(t)=\sqrt{t}$, $y(t)=1/t$. Express the final result in terms of t.

Solution:

$$\begin{aligned}\frac{df}{dt}&=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=(\cos x\cos y)\left(\frac{1}{2}t^{-1/2}\right)-(\sin x\sin y)\left(-t^{-2}\right)\\&=\frac{1}{2\sqrt{t}}\cos\sqrt{t}\,\cos\frac{1}{t}+\frac{1}{t^2}\sin\sqrt{t}\,\sin\frac{1}{t}\end{aligned}$$

(c) Compute the partial derivatives of f with respect to x,y for $f(s,t)=s^2-t^2, s=\dfrac{x+y}{2}, t=\dfrac{x-y}{2}$ using both the chain rule and by plugging in s and t.

Solution: Using the chain rule:

$$\begin{aligned}\frac{\partial f}{\partial x}&=\frac{\partial f}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial f}{\partial t}\frac{\partial t}{\partial x}\\&=2s\cdot\frac{1}{2}-2t\cdot\frac{1}{2}=s-t=y\\\frac{\partial f}{\partial y}&=\frac{\partial f}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial f}{\partial t}\frac{\partial t}{\partial y}\\&=2s\cdot\frac{1}{2}+2t\cdot\frac{1}{2}=s+t=x\end{aligned}$$

Plugging in we see $f(s(x,y),t(x,y))=\frac{1}{4}((x+y)^2-(x-y)^2)=xy$ and easily obtain the same partial derivatives.

(d) Compute the partial derivatives of $f(x,y)=\left(\sqrt{x^2+y^2}-\frac{1}{\sqrt{x^2+y^2}}\right)e^{\frac{x^2+y^2}{2}}$ by identifying an appropriate expression $r(x,y)$ such that $f(r)$ is a simpler expression. You can assume $(x,y)\neq(0,0)$. It's fine to have r in the final expression.

Solution: It is useful to introduce $r = \sqrt{x^2+y^2}$ such that $f(r) = (r + 1/r)\, e^{-r^2/2}$. It's not hard to see that $f'(r) = -(r^2 + r^{-2})e^{-r^2/2}$ and $\frac{\partial r}{\partial x} = x/r$ and $\frac{\partial r}{\partial y} = y/r$. Using the chain rule we now get

$$f_x(x,y) = -(r + r^{-3})e^{-r^2/2}x \quad (1)$$

$$f_y(x,y) = -(r + r^{-3})e^{-r^2/2}y \quad (2)$$

(e) Suppose z is given by $z = e^{s+2t}$, where $s = x/y$ and $t = y/x$. Compute $\partial z/\partial x$ and $\partial z/\partial y$. Express the result in x, y and z.

Solution:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x} = z/y - 2yz/x^2$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial y} = -xz/y^2 + 2z/x$$

2 Challenge

Consider the function

$$f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that $f_{xy}(0,0)$ and $f_{yx}(0,0)$ but exist, but are unequal. Warning: Since the second partial derivatives are discontinuous at $(0,0)$, you will not get the right answer by evaluating $f_{xy}(a,b)$ and $f_{yx}(a,b)$ and taking the limit for $(a,b) \to (0,0)$.

Solution:
We have to use the definition of partial derivatives to find the mixed derivatives, for example: $f_{xy}(0,0) = \lim_{y\to 0} \frac{f_x(0,y) - f_x(0,0)}{y}$
First of all,

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}.$$

We substitute $x = 0$ in the above to find $f_x(0,y) = -y$ when $y \neq 0$.
Also, $f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(x)}{x} = 0$ since f is zero.
Hence, altogether, $f_{xy}(0,0) = \lim_{y\to 0} \frac{-y}{y} = -1$.
Working similarly (or arguing by symmetry, noting that the function is antisymmetric in x, y) for the other mixed derivative, we will find that the answer is $+1$ instead.

3 True/False

(a) T F Consider $f(x,y) = (\cos x + (1+x)\tan y)e^{x^2-1+y}$. Then $\frac{d}{dt}f(t^2, t^3) = 0$ at $t = 0$.

Solution: **TRUE.** Writing $x(t) = t^2$ and $y(t) = t^3$ we see that $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ at $t = 0$. Now we see from the chain rule that $\frac{d}{dt}f(x(t), y(t)) = 0$ at $t = 0$.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.