Discussion 6 Worksheet Answers
Tangent Planes and Linear Approximations

Date: 9/13/2021

MATH 53 Multivariable Calculus

1 Tangent Planes

Find the tangent planes to the graphs of each of the following functions at an arbitrary point $(x_0, y_0, f(x_0, y_0))$.

1. $f(x, y) = x^2 + 2xy + y^2$

   Solution: We have $f_x = 2x + 2y$ and $f_y = 2x + 2y$, so plugging these into the tangent plane equation

   $$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

   we get

   $$z - x_0^2 - 2x_0y_0 - y_0^2 = (2x_0 + 2y_0)(x - x_0) + (2x_0 + 2y_0)(y - y_0).$$

2. $f(x, y) = e^{xy}$.

   Solution: We have $f_x = ye^{xy}$ and $f_y = xe^{xy}$, so plugging this into the tangent plane equation (above), we get

   $$z - e^{x_0y_0} = y_0e^{x_0y_0}(x - x_0) + x_0e^{x_0y_0}(x - x_0).$$

3. $f(x, y) = \sin x$.

   Solution: We have $f_x = \cos x$ and $f_y = 0$, so the tangent plane equation is

   $$z - \sin x_0 = (\cos x_0)(x - x_0).$$

2 More Tangent Planes

1. Find an equation for the tangent plane to the graph of $f(x, y) = \cos(xy)$ passing through the point $(\pi/2, 1, 0)$.

Solution: We have $f_x = -y\sin(xy)$ and $f_y = -x\sin(xy)$, so the tangent plane at a general point (x_0, y_0) is given by

$$z - \cos(x_0 y_0) = -y_0 \sin(x_0 y_0)(x - x_0) - x_0 \sin(x_0 y_0)(y - y_0).$$

Plugging in our point $(\pi/2, 1, 0)$, this becomes

$$z = -1 \cdot \left(x - \frac{\pi}{2}\right) - \frac{\pi}{2} \cdot (y - 1),$$

or equivalently

$$x + \frac{\pi}{2}y + z = \pi.$$

2. Find a parametric equation for a line contained in the tangent plane you found in the previous problem. (Any line will suffice.)

Solution: Note that the point $(\pi/2, 1, 0)$ lies on this plane, and the plane has normal vector $\langle 1, \pi/2, 1\rangle$. Any vector orthogonal to the normal vector will point along this plane. By inspection, we see that $\langle 1, 0, -1\rangle \cdot \langle 1, \pi/2, 1\rangle = 0$, so the vector $\langle 1, 0, -1\rangle$ points along the plane. Thus $\langle 1, 0, -1\rangle$ is the direction vector of a line pointing along this plane. Combining this with the point on the plane that we have already found, we see that the line

$$\vec{r}(t) = \left\langle t + \frac{\pi}{2}, 1, -t \right\rangle$$

is contained in the tangent plane. We could find other lines contained in the plane by making different choices of direction vector.

3 Linear Approximations

1. Find the best linear approximation to each of the following functions near the corresponding input values.

a) $f(x, y) = y^2 - x$ near the input $(3, 0)$.

Solution: We have $f_x(x, y) = -1$ and $f_y(x, y) = 2y$, so $f_x(3, 0) = -1$ and $f_y(3, 0) = 0$. Our formula for the best linear approximation near $(3, 0)$ is

$$f(x, y) \approx f(3, 0) + f_x(3, 0) \cdot (x - 3) + f_y(3, 0) \cdot (y - 0),$$

so we see

$$f(x, y) \approx -3 - (x - 3) = -x$$

for (x, y) near $(3, 0)$.

b) $g(x, y) = e^x \cos y$ near the input $(5, \pi/2)$.

Solution: We have $g_x(x, y) = e^x \cos y$ and $g_y(x, y) = -e^x \sin y$, so $g_x(5, \pi/2) = 0$ and $g_y(5, \pi/2) = -e^5$. We also compute $g(5, \pi/2) = 0$. Plugging these into our formula for the best linear approximation gives

$$g(x, y) \approx -e^5 \left(y - \frac{\pi}{2}\right).$$

c) $h(x, y, z) = xyz$ near the input $(3, 0, 2)$.

Solution: We have $h_x(x, y, z) = yz$, $h_y(x, y, z) = xz$, and $h_z(x, y, z) = xy$. Thus, at the input $(3, 0, 2)$, we have $h = 0$, $h_x = 0$, $h_y = 6$, and $h_z = 0$. Using a formula similar to that for the 2-dimensional case, we see that the best linear approximation is

$$h(x, y, z) \approx 6(y - 0) = 6y.$$

d) $p(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ near the input $(0, 1, 0, -1)$.

Solution: Even though this is a function of four variables, our old methods still work! We just have to add a few more terms to our sums to account for the extra variables. At the input $(0, 1, 0, -1)$, we have $p = 2$, $p_x = 0$, $p_y = 2$, $p_z = 0$, and $p_w = -2$. Thus the best linear approximation is given by

$$p(x, y, z, w) \approx 2 + 2(y - 1) - 2(w + 1) = -2 + 2y - 2w.$$

2. Consider a differentiable function $f(x, y)$ with values given by the following table.

	$x = 1.0$	$x = 1.2$
$y = 0.0$	5.2	5.4
$y = 0.2$	6.0	6.2

a) Find the best linear approximation to $f(x, y)$ near the input value $(1.0, 0.0)$.

Solution: We approximate

$$\frac{\partial f}{\partial x}(1.0, 0.0) \approx \frac{f(1.2, 0.0) - f(1.0, 0.0)}{0.2} = \frac{5.4 - 5.2}{0.2} = 1$$

and

$$\frac{\partial f}{\partial y}(1.0, 0.0) \approx \frac{f(1.0, 0.2) - f(1.0, 0.0)}{0.2} = \frac{6.0 - 5.2}{0.2} = 4.$$

Furthermore, we have $f(1.0, 0.0) = 5.2$. So our best linear approximation is given by

$$\begin{aligned} f(x, y) &\approx f(1.0, 0.0) + \frac{\partial f}{\partial x}(1.0, 0.0)(x - 1.0) + \frac{\partial f}{\partial y}(1.0, 0.0)(y - 0.0) \\ &\approx 5.2 + x - 1.0 + 4y \\ &= 4.2 + x + 4y. \end{aligned}$$

b) Use this linear approximation to compute approximate values for $f(1.0, 0.1)$, $f(1.1, 0.0)$ and $f(1.1, 0.1)$.

Solution: We just plug in the given values into our result from the first part. For example, we have

$$f(1.0, 0.1) \approx 4.2 + 1.0 + 4 \cdot 0.1 = 5.6.$$

Similarly, we see $f(1.1, 0.0) \approx 5.3$ and $f(1.1, 0.0) \approx 5.7$.

4 Implicit differentiation

1. Find $\partial z/\partial x$, $\partial z/\partial y$ and $\partial x/\partial y$ when x, y and z satisfy the relation $x^2 + y^2 + z^2 = 3xyz$.

Solution: We can write the equation as $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz = 0$. We compute

$$\frac{\partial F}{\partial x} = 2x - 3yz$$
$$\frac{\partial F}{\partial y} = 2y - 3xz$$
$$\frac{\partial F}{\partial z} = 2z - 3xy$$

Using the formulas $\frac{\partial x}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial x}$ etc.

$$\frac{\partial x}{\partial y} = -\frac{2y - 3xz}{2x - 3yz}$$
$$\frac{\partial y}{\partial z} = -\frac{2z - 3xy}{2y - 3xz}$$
$$\frac{\partial z}{\partial x} = -\frac{2x - 3yz}{2z - 3xy}$$

2. (Challenge) Suppose that x, y, z are related by an equation $F(x, y, z) = 0$ (this is the setup for implicit differentiation). Show that

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$$

Solution: From the definition $\frac{\partial x}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial x}$ and analogous for other variables. So we immediately see

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = \left(-\frac{\partial F/\partial y}{\partial F/\partial x}\right)\left(-\frac{\partial F/\partial z}{\partial F/\partial y}\right)\left(-\frac{\partial F/\partial x}{\partial F/\partial z}\right) = -1$$

5 Challenge

1. Let S be a sphere centered at the origin in $\mathbb{R}^3$, and consider any point P on S. Show that the vector $\overrightarrow{OP}$ is orthogonal to the tangent plane to S at P.

Solution: Let R be the radius of the sphere, and write $P = (x_0, y_0, z_0)$. Assume for the moment that P lies on the top half of the sphere (i.e. $z_0 > 0$). Note that the top half of the sphere is the same as the graph of the function $f(x, y) = \sqrt{R^2 - x^2 - y^2}$. We can compute

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) = -\frac{x_0}{\sqrt{R^2 - x_0^2 - y_0^2}}$$

and

$$\frac{\partial f}{\partial y}(x_0, y_0, z_0) = -\frac{y_0}{\sqrt{R^2 - x_0^2 - y_0^2}},$$

so that the tangent plane to the graph at P (which is the same as the tangent plane to S at P) is given by

$$z - z_0 = -\frac{x_0}{\sqrt{R^2 - x_0^2 - y_0^2}}(x - x_0) - \frac{y_0}{\sqrt{R^2 - x_0^2 - y_0^2}}(y - y_0).$$

We can rewrite this as

$$x_0(x - x_0) + y_0(y - y_0) + z_0(z - z_0) = 0,$$

where we use the fact that $z_0 = \sqrt{R^2 - x_0^2 - y_0^2}$ to simplify things. From this we see that $\langle x_0, y_0, z_0\rangle$ is a normal vector to the tangent plane. But $\langle x_0, y_0, z_0\rangle = \overrightarrow{OP}$, so this is exactly what we needed to show.

A similar argument (with $-\sqrt{R^2 - x^2 - y^2}$ in place of $\sqrt{R^2 - x^2 - y^2}$) works when P is on the bottom half of the sphere (i.e. $z_0 < 0$). When $z = 0$, we have to use a "sideways graph" of some function like $f(y, z) = \sqrt{R^2 - y^2 - z^2}$, but other than that, pretty much everything is the same.

6 True/False

Supply convincing reasoning for your answer.

(a) T F The vector $\langle f_x(x_0, y_0), f_y(x_0, y_0), -1\rangle$ is orthogonal to the tangent plane of the graph of $f(x, y)$ through the point (x_0, y_0, z_0).

Solution: TRUE. This tangent plane is given by the equation

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

and rewriting this equation as

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0,$$

we see that a normal vector to this plane is given by $\langle f_x(x_0, y_0), f_y(x_0, y_0), -1\rangle$.

(b) T F Any tangent plane to a graph must meet that graph in exactly one point.

Solution: FALSE. The tangent plane could meet the graph in many other points. For example, the tangent plane to the graph of the constant function $f(x, y) = 1$ is the same as the graph itself, so meets the graph in infinitely many points.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.