Discussion 5 Worksheet Answers

Vector-valued functions and partial derivatives

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MATH 53 Multivariable Calculus

1 Integrals of Vector Functions

1. Evaluate $\int_1^2 \langle \frac{\ln(t)}{t}, e^{-t} \rangle dt$.

   Solution: In the first component, $\int_1^2 \frac{\ln(t)}{t} dt = \int_0^{\ln(2)} u du = \frac{1}{2} \ln^2(2)$ (with the substitution $u = \ln(t)$.
   In the second component, $\int_1^2 e^{-t} dt = e^{-1} - e^{-2}$.
   So, altogether, we get $\langle \frac{1}{2} \ln^2(2), \frac{1}{e} - \frac{1}{e^2} \rangle$.

2. Suppose that $\vec{r''}(t) = \langle 6t, \sin(t) \rangle$ and it is known that $\vec{r'}(0) = \langle 1, -1 \rangle$ and $\vec{r}(0) = \langle 0, 1 \rangle$. Find a formula for $\vec{r}(t)$.

   Solution:
   We integrate: $\vec{r'}(t) = \int_0^t \vec{r''}(s) ds + \vec{r'}(0) = \int_0^t \langle 6t, \sin(t) \rangle dt + \langle 1, -1 \rangle = \langle 3t^2 + 1, -\cos(t) \rangle$, and by the same process, $\vec{r}(t) = \langle t^3 + t, -\sin(t) + 1 \rangle$.

2 Vector Function Basics

(a) Find the limit

$$\lim_{t \to 0} \left\langle e^{-3t}, \frac{t^2}{\sin^2 t}, \cos 2t \right\rangle.$$

Solution: By applying L'Hospital's rule twice , we obtain $\langle 1, 1, 1 \rangle$.

(b) Find the limit

$$\lim_{t \to \infty} \left\langle \frac{1 + t^2}{1 - t^2}, \arctan t, \frac{1 - e^{-2t}}{t} \right\rangle.$$

Solution: By applying L'Hospital's rule twice , we obtain $\langle -1, \frac{\pi}{2}, 0 \rangle$.

(c) Find a vector equation and parametric equations for the line segment that joins $(2, 0, 0)$ to $(6, 2, -2)$.

Solution: We have $\mathbf{r}(t) = (1 - t)\langle 2, 0, 0 \rangle + t\langle 6, 2, -2 \rangle, 0 \le t \le 1$ which simplifies to $\mathbf{r}(t) = \langle 2 + 4t, 2t, -2t \rangle, 0 \le t \le 1$. The parametric equations are $x = 4 + 2t, y = 2t, z = -2t, 0 \le t \le 1$.

(d) Find a vector equation and parametric equations for the line segment that joins $(1, 5, 6)$ to $(3, 1, 8)$.

Solution: We have $\mathbf{r}(t) = (1 - t)\langle 1, 5, 6 \rangle + t\langle 3, 1, 8 \rangle, 0 \le t \le 1$ which simplifies to $\mathbf{r}(t) = \langle 1 + 2t, 5 - 4t, 6 + 2t \rangle, 0 \le t \le 1$. The parametric equations are $x = 1 + 2t, y = 5 - 4t, z = 6 + 2t, 0 \le t \le 1$.

(e) Find a vector function that represents the curve of the intersection of the cone $z = \sqrt{x^2 + y^2}$ and $z = 1 + y$.

Solution: We equate the two equations to obtain $\sqrt{x^2 + y^2} = 1 + y$ so $x^2 + y^2 = 1 + 2y + y^2$. Simplifying we get $y = \frac{1}{2}(x^2 - 1)$. We can form parametric equations for the curve of intersection C by setting $x = t$. Then $y = \frac{1}{2}(t^2 - 1)$ and $z = 1 + y = \frac{1}{2}(t^2 + 1)$. Hence, the vector function representing C is $\mathbf{r}(t) = \langle t, \frac{1}{2}(t^2 - 1), \frac{1}{2}(t^2 + 1)\rangle$.

(f) Suppose the trajectories of two particles are given by

$$r_1(t) = \langle t^2, 7t - 12, t^2\rangle \qquad r_2(t) = \langle 4t - 3, t^2, 5t - 6\rangle$$

for $t \geq 0$. Do the particles collide?

Solution: For the particles to collide, we need $r_1(t) = r_2(t)$. Equating components gives us $t^2 = 4t - 3, 7t - 12 = t^2, t^2 = 5t - 6$. From the first equation, $t^2 - 4t + 3 = 0$ has solutions at $t = 1, 3$. We see that $t = 1$ does not satisfy the other two equations but $t = 3$ does so the particles collide at $t = 3$ at the point $(9, 9, 9)$.

3 Challenge: Vector Orthogonality

(a) Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Solution:

(a) Since $\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$. Then taking derivatives yields,

$$\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

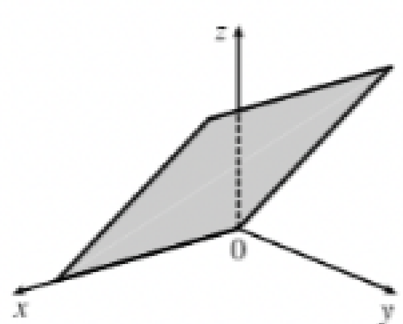
so $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$ so $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

4 Graphs of Multivariable Functions

Sketch the graph of the function.

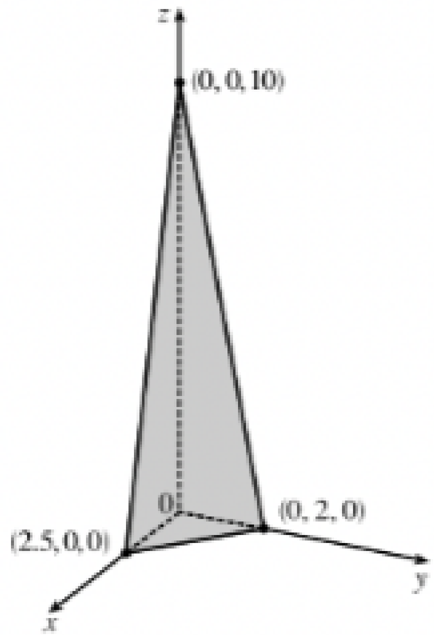
(a) $f(x, y) = y$;

Solution: The graph of f has equation $z = y$ which can be though of as a line in the yz-plane and then extending that line across the x-axis giving us a plane.



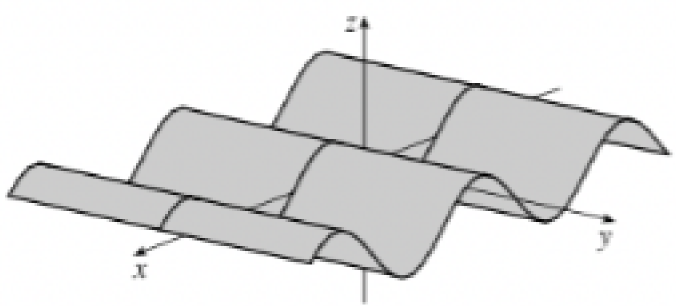
(b) $f(x, y) = 10 - 4x - 5y$;

Solution: This is $z = 10 - 4x - 5y$ so it is a plane with intercepts $(2.5, 0, 0), (0, 2, 0), (0, 0, 10)$.



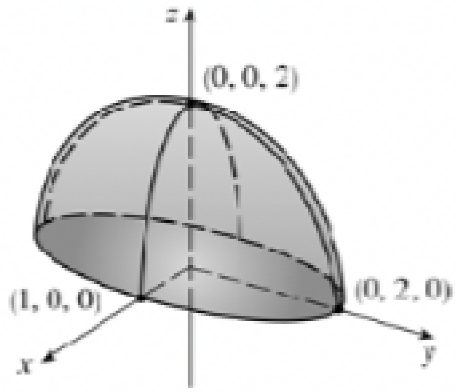
(c) $f(x, y) = \sin x$;

Solution: The graph of f has equation $z = \sin x$ so we first think of this as the regular sin graph and then extend it along the entire y-axis.



(d) $f(x, y) = \sqrt{4 - 4x^2 - y^2}$.

Solution: This is $z = \sqrt{4 - 4x^2 - y^2}$ so this is the top half of the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{4} = 1$.



5 Evaluating Partial Derivatives

1. Compute the following partial derivatives:

   $\frac{\partial}{\partial x}(x^2 e^{xy})$.

   Solution: Applying the product rule, the answer is $2xe^{xy} + x^2 y e^{xy}$.

   $\frac{\partial^{10}}{\partial x^{10}}\left(\frac{\partial^{13}}{\partial y^{13}}\left(x^{10}y^{13}\right)\right)$.

   Solution: Since $\frac{d^n}{dt^n} t^n = n!$ in general, in our case, we get 10!13!.

   $\frac{\partial}{\partial w}(\sin(w \sin(wv)))$.

   Solution: Following the chain rule, the answer is $(\sin(wv) + wv\cos(wv))\cos(w\sin(wv))$.

   $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{e^{xy\sin(y)-x}}{y\sin(y)-1}\right)\right)$.Hint: Clairaut's Theorem simplifies the calculation.

   Solution: Differentiating with respect to x first gives

   $$\frac{y\sin(y) - 1}{y\sin(y) - 1} e^{xy\sin(y)-x} = e^{xy\sin(y)-x}.$$

   Then differentiating with respect to y gives $(x\sin(y) + xy\cos(y)))\, e^{xy\sin(y)-x}$.

2. In the equation $PV = T$, any one of the three variables can be solved for as a function of the other two. Show that $\left(\frac{\partial P}{\partial T}\right)\left(\frac{\partial T}{\partial V}\right)\left(\frac{\partial V}{\partial P}\right) = -1$.

   Solution: This product is $\left(\frac{1}{V}\right)(P)\left(\frac{-T}{P^2}\right) = \frac{-T}{PV} = -1$.

6 More on Partial Derivatives

1. Suppose that the values of a function $f(x, y)$ at four points are given by the following table:

	x=1.3	x=1.4
y=0.4	2.3	2.5
y=0.6	1.5	1.4

Estimate $f_x(1.3, 0.4)$ and $f_x(1.3, 0.6)$. Then estimate $f_{xy}(1.3, 0.4)$.

Solution: First of all $f_x(1.3, 0.4) \approx \frac{f(1.4,0.4)-f(1.3,0.4)}{(0.1)} = \frac{0.2}{0.1} = 2$. Also, $f_x(1.3, 0.6) \approx \frac{f(1.4,0.6)-f(1.3,0.6)}{0.1} = \frac{-0.1}{0.1} = -1$.
Next, $f_{xy}(1.3, 0.4) \approx \frac{f_x(1.3,0.6)-f_x(1.3,0.4)}{0.2} = \frac{-3}{0.2} = -15$.

2. Consider a smooth function $f(x, y)$ of two variables. List all possible third order partial derivatives of f. Your list should not contain two equivalent expressions. Here "smooth" means that all relevant derivatives of f exist and are continuous.

Solution: By Clairaut's theorem, only the total number of derivatives with respect to x and y are relevant, since order does not matter. The possibilities are $f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy}$.

3. Suppose that the partial derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}$ exist. If f_x is the zero function, show that f is a function of y only. More precisely, there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(y)$ for all real numbers x and y.

Hint: For fixed y, consider the function $h(t) = f(t, y)$ and differentiate.

Solution: As suggested, consider the function $h(t) = f(t, y)$, Then $h'(t) = f_x(t, y) = 0$, so h is a constant function. This means (for example) $h(x) = h(0)$ for all x, so $f(x, y) = f(0, y)$ is a function of y only.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.