Discussion 4 Worksheet Answers
Vectors
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MATH 53 Multivariable Calculus

1 Dot Products

1. If $\vec{u}$ and $\vec{v}$ are unit vectors in $\mathbb{R}^3$ and $u \circ v = -1$, what is the angle between $\vec{u}$ and $\vec{v}$?

Solution: From the formula $\vec{u} \circ \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos\theta$, it follows that $\cos\theta = -1$, so $\theta = \pi$. This may also be seen by showing directly that $|\vec{u} + \vec{v}|^2 = 0$, so $\vec{u} = -\vec{v}$.

2. Find three nonzero vectors in $\mathbb{R}^3$ that are perpendicular to $\langle 1, 3, 2\rangle$.

Solution: A nonzero vector $\langle x, y, z\rangle$ will work if and only if $x+3y+2z = 0$. Specifically $\langle -1, 1, -1\rangle$ and $\langle 2, 0, -1\rangle$, and $\langle 3, -1, 0\rangle$ all work (alternatively, once one solution is found, it may be scaled to find others).

3. Let P be a vertex on a cube. Let Q be an adjacent vertex and let R be the vertex opposite to P. Using dot products, find the angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

Solution: Without loss of generality, take the cubic to lie in the first octant, with edges along the positive coordinate axes, and have edges of length 1, so that $P = (0, 0, 0)$ and $Q = (1, 0, 0)$. Then $R = (1, 1, 1)$ and $\overrightarrow{PQ} = \langle 1, 0, 0\rangle$. Similarly $\overrightarrow{PR} = \langle 1, 1, 1\rangle$, so $\overrightarrow{PQ} \circ \overrightarrow{PR} = 1 = |\overrightarrow{PQ}| \cdot |\overrightarrow{PR}| \cos(\theta) = \sqrt{3}$, so $\theta = \arccos(1/\sqrt{3})$

4. If $\vec{u}$ and $\vec{v}$ are unit vectors in $\mathbb{R}^3$, show that the vectors $\vec{u} + \vec{v}$ and $\vec{v} - \vec{v}$ are perpendicular.

Solution: We have $(\vec{u} + \vec{v}) \circ (\vec{u} - \vec{v}) = |\vec{u}|^2 - |\vec{v}|^2 = 1 - 1 = 0$.

5. Derive the *polarization identity*: if $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$, then $\vec{u} \circ \vec{v} = \frac{1}{4}\left(|\vec{u} + \vec{v}|^2 - |\vec{u} - \vec{v}|^2\right)$. Hint: it is simplest not to work straight from the definition of the dot product (although this will work too).

Solution: This follows from writing $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \circ (\vec{u} + \vec{v})$ (and similarly for the other term) and expanding.

2 Challenge: Parallelogram Law

Consider a parallelogram with side lengths a and b, and diagonals of lengths c and d. Show that $2a^2 + 2b^2 = c^2 + d^2$. Hint: use vector geometry and dot products.

Solution: Let P, Q, R, S be the vertices of the parallelogram, listed in cyclic order so that $a = |\overrightarrow{PQ}|$, etc. Then $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$. Taking length squared of both sides gives

$$|\overrightarrow{PR}|^2 = |\overrightarrow{PQ}|^2 + |\overrightarrow{QR}|^2 + 2\overrightarrow{PQ} \circ \overrightarrow{QR}.$$

Similarly,

$$|\overrightarrow{RS}|^2 = |\overrightarrow{PQ}|^2 + |\overrightarrow{QR}|^2 - 2\overrightarrow{PQ} \circ \overrightarrow{QR}.$$

Then add to get the desired result.

3 Vector and Scalar Projections

1. For each of the following pairs of vectors, find the vector projection of $\vec{v}$ onto $\vec{w}$ and the scalar projection of $\vec{v}$ onto $\vec{w}$.

   a) $\vec{v} = \langle 2, 4\rangle$, $\vec{w} = \langle 3, 1\rangle$.

   Solution: The vector projection is

   $$\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w} = \frac{2\cdot 3 + 4\cdot 1}{3^2+1^2}\langle 3,1\rangle = \frac{10}{10}\langle 3,1\rangle = \langle 3,1\rangle,$$

   and the scalar projection is

   $$\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|} = \frac{2\cdot 3+4\cdot 1}{\sqrt{3^2+1^2}} = \frac{10}{\sqrt{10}} = \sqrt{10}.$$

   b) $\vec{v} = \langle 5, -1\rangle$, $\vec{w} = \langle 2, 9\rangle$.

   Solution: The vector projection is

   $$\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w} = \frac{5\cdot 2 - 1\cdot 9}{2^2+9^2}\langle 2,9\rangle = \frac{1}{85}\langle 2,9\rangle = \left\langle \frac{2}{85}, \frac{9}{85}\right\rangle,$$

   and the scalar projection is

   $$\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|} = \frac{5\cdot 2 - 1\cdot 9}{\sqrt{2^2+9^2}} = \frac{2}{\sqrt{85}}.$$

   c) $\vec{v} = \langle -6, 3, 2\rangle$, $\vec{w} = \langle 1, -5, 3\rangle$.

   Solution: The vector projection is

   $$\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w} = \frac{-6\cdot 1 - 3\cdot 5 + 2\cdot 3}{1^2+5^2+3^2}\langle 1,-5,3\rangle = -\frac{3}{7}\langle 1,-5,3\rangle = \left\langle -\frac{3}{7}, \frac{15}{7}, \frac{9}{7}\right\rangle,$$

   and the scalar projection is

   $$\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|} = \frac{-6\cdot 1 - 3\cdot 5 + 2\cdot 3}{\sqrt{1^2+5^2+3^2}} = -3\sqrt{\frac{5}{7}}.$$

2. Find formulas for the vector and scalar projections of a vector $\vec{v}$ onto $\vec{w}$ involving the cosine of the angle θ between $\vec{v}$ and $\vec{w}$.

Solution: Recall that we can write the dot product using

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta.$$

Substituting this into our formulas for vector and scalar projections, we see that the vector projection is

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2}\vec{w} = \frac{|\vec{v}||\vec{w}| \cos\theta}{|\vec{w}|^2}\vec{w} = \frac{|\vec{v}|}{|\vec{w}|} \cos\theta \vec{w}$$

and the scalar projection is

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} = \frac{|\vec{v}||\vec{w}| \cos\theta}{|\vec{w}|} = |\vec{v}| \cos\theta.$$

Both of these formulas can also be derived from thinking hard enough about the geometry of the situation.

4 Cross Product Computations

Find the cross products $\vec{v} \times \vec{w}$ of the following pairs of vectors.

1. $\vec{v} = \langle 2, 3, 1\rangle$, $\vec{w} = \langle -1, 2, 3\rangle$.

Solution: We use the determinant formula:

$$\begin{aligned}
\vec{v} \times \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ -1 & 2 & 3 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \vec{k} \\
&= (3^2 - 1 \cdot 2)\vec{i} - (2 \cdot 3 - 1 \cdot (-1))\vec{j} + (2 \cdot 2 - 3 \cdot (-1))\vec{k} \\
&= 7\vec{i} - 8\vec{j} + 7\vec{k}.
\end{aligned}$$

2. $\vec{v} = 6\vec{i} - 4\vec{j} - 3\vec{j}$, $\vec{w} = 4\vec{i} + \vec{j}$.

Solution: We use the determinant formula:

$$\begin{aligned}
\vec{v} \times \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -4 & -3 \\ 4 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 6 & -3 \\ 4 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 6 & -4 \\ 4 & 1 \end{vmatrix} \vec{k} \\
&= 3\vec{i} - 12\vec{j} + 22\vec{k}.
\end{aligned}$$

3. $\vec{v}$ pointing a distance 5 units in the positive x-direction, $\vec{w}$ a unit vector lying in the first quadrant of the xy-plane and making an angle of $\pi/4$ with the x-axis.

Solution: The right-hand rule tells us that the cross product will point in the positive z direction. To compute its magnitude, we use the formula

$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta.$$

Here $|\vec{v}| = 5$, $|\vec{w}| = 1$, and $\theta = \pi/4$, so we obtain

$$|\vec{v} \times \vec{w}| = 5 \sin\frac{\pi}{4} = \frac{5\sqrt{2}}{2},$$

and thus

$$\vec{v} \times \vec{w} = \frac{5\sqrt{2}}{2}\vec{k}.$$

5 Cross Product Concepts and Applications

1. Given vectors $\vec{v}$ and $\vec{w}$, find an identity which relates the four quantities $|\vec{v}|$, $|\vec{w}|$, $|\vec{v} \times \vec{w}|$, and $|\vec{v} \cdot \vec{w}|$. (Hint: Consider any relevant trigonometric identities.)

Solution: If θ is the angle between the vectors, we can write

$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$$

and

$$|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}| \cos\theta.$$

Since $\cos^2\theta + \sin^2\theta = 1$, we can square and add the above equations to get

$$|\vec{v} \times \vec{w}|^2 + |\vec{v} \cdot \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 \sin^2\theta + |\vec{v}|^2|\vec{w}|^2 \cos^2\theta = |\vec{v}|^2|\vec{w}|^2.$$

Thus our identity is

$$|\vec{v} \times \vec{w}|^2 + |\vec{v} \cdot \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2.$$

2. Let $\vec{u}$ and $\vec{v}$ be nonzero vectors with $\vec{u} \times \vec{v} = \vec{0}$. What can you say about the relationship between $\vec{u}$ and $\vec{v}$?

Solution: Let θ be the angle between the two vectors; then we have

$$0 = |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta.$$

This can only happen if $\sin\theta = 0$, which implies that θ is an integer multiple of π. Thus we may conclude that $\vec{u}$ and $\vec{v}$ are collinear.

3. Find the area of the triangle with two sides given by the vectors $\vec{v} = \langle 1, 2\rangle$ and $\vec{w} = \langle -3, 4\rangle$.

Solution: We view this triangle as sitting within the xy-plane in $\mathbb{R}^3$. Then the quantity $|\vec{v} \times \vec{w}|$ gives the area of the parallelogram with two sides given by $\vec{v}$ and $\vec{w}$. We compute

$$\vec{v} \times \vec{w} = \langle 0, 0, 1 \cdot 4 - 2 \cdot (-3)\rangle = 10\vec{k},$$

where we are justified in ignoring the $\vec{i}$ and $\vec{j}$ components because we know that $\vec{v} \times \vec{w}$ must be orthogonal to the xy-plane. So the area of this parallelogram is 10. The area of the triangle is half that of the parallelogram, so we see that the desired area is 5.

6 Challenge: BAC-CAB

Prove the "BAC-CAB" / "double-crossing" rule:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}).$$

(NOTE: Typically scalars such as $\vec{a} \cdot \vec{c}$ are written on the left in scalar multiplication. This formula is a rare exception and is written this way because "BAC-CAB" is easier to remember than "ACB-ABC.")

Solution: We can write

$$\vec{b} \times \vec{c} = (b_2c_3 - b_3c_2)\vec{i} + (b_3c_1 - b_1c_3)\vec{j} + (b_1c_2 - b_2c_1)\vec{k},$$

so

$$\begin{aligned}\vec{a} \times (\vec{b} \times \vec{c}) =& (a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3))\vec{i} + \\ & (a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1))\vec{j} + \\ & (a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2))\vec{k}.\end{aligned}$$

Rearranging this gives

$$\begin{aligned}\vec{a} \times (\vec{b} \times \vec{c}) =& (b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3))\vec{i} + \\ & (b_2(a_3c_3 + a_1c_1) - c_2(a_3b_3 + a_1b_1))\vec{j} + \\ & (b_3(a_1c_1 + a_2c_2) - c_3(a_1b_1 + a_2b_2))\vec{k},\end{aligned}$$

and adding / subtracting a copy of $a_1b_1c_1$ from the first component, $a_2b_2c_2$ from the second component, and $a_3b_3c_3$ from the last component gives

$$\begin{aligned}\vec{a} \times (\vec{b} \times \vec{c}) =& (b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3))\vec{i} + \\ & (b_2(a_2c_2 + a_3c_3 + a_1c_1) - c_2(a_2b_2 + a_3b_3 + a_1b_1))\vec{j} + \\ & (b_3(a_3c_3 + a_1c_1 + a_2c_2) - c_3(a_3b_3 + a_1b_1 + a_2b_2))\vec{k},\end{aligned}$$

which can be written more succinctly as

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}).$$

7 True/False

Supply convincing reasoning for your answer.

(a) T F If you take a cross product of two vectors lying in the xy-plane, your result will point along the z-axis.

Solution: TRUE. The cross product produces a result that is orthogonal to both inputs, and the z-axis is the space of vectors orthogonal to the xy-plane.

(b) T F The cross product makes sense for vectors in any number of dimensions.

Solution: FALSE. The cross product only makes sense for vectors in $\mathbb{R}^3$ (and technically $\mathbb{R}^7$, though people don't really use that much). The formula you use to define the cross product doesn't adapt well to other numbers of dimensions.

(c) T F The absolute value of the scalar projection of a vector $\vec{v}$ onto another vector $\vec{w}$ is equal to the norm of the vector projection of $\vec{v}$ onto $\vec{w}$.

Solution: TRUE. The vector projection is given by

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w},$$

and taking the norm of this gives

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} |\vec{w}| = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|},$$

which is just the absolute value of the scalar projection.

(d) T F The cross product is associative: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$.

Solution: FALSE. For example, we have

$$(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0},$$

but

$$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}.$$

(e) T F The dot and cross products satisfy $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b}) \times \vec{c}$.

Solution: FALSE. The right-hand side of the purported equation is not even well-defined, as $\vec{a} \cdot \vec{b}$ is a scalar, and you cannot take the cross product of a scalar and a vector.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.