Discussion 3 Worksheet Answers

Polar coordinates

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MATH 53 Multivariable Calculus

1 Computing Tangents to Polar Curves

Compute the slopes of the following curves. Find the points where the tangents are vertical and horizontal.

(a) $r = 3\cos\theta$;

(b) $r = 1 - \sin\theta$;

(c) $r = \sec\theta$;

(d) $r = e^{\theta}$.

Solution:

(a) $dy/d\theta = -3\sin^2\theta + 3\cos^2\theta$ and $dx/d\theta = -6\sin\theta\cos\theta$ so the slope is $\frac{dy}{dx} = \frac{3\cos(2\theta)}{-3\sin(2\theta)} = -\tan(2\theta)$. The horizontal tangents are where $3\cos(2\theta) = 0$ which occurs at $\pi/4 + k\pi/2, k \in \mathbb{Z}$ and the vertical tangents are where $-3\sin(2\theta) = 0$ which occurs at $k\pi/2, k \in \mathbb{Z}$.

(b) $dy/d\theta = \cos\theta - \sin(2\theta)$ and $dx/d\theta = -\sin\theta - \cos(2\theta)$ so the slope is $\frac{dy}{dx} = \frac{\cos\theta - \sin(2\theta)}{-\sin\theta - \cos(2\theta)}$. The horizontal tangents are where $\cos\theta(1 - 2\sin\theta) = 0$ which occurs at $\theta = \pi/2 + k\pi, k \in \mathbb{Z}$ and $\theta = \pi/6 + 2k\pi, 5\pi/6 + 2l\pi, k, l \in \mathbb{Z}$. The vertical tangents are where $1 + \sin\theta - 2\sin^2\theta = (1 + 2\sin\theta)(1 - \sin\theta) = 0$ which occurs at $\pi/2 + 2k\pi, k \in \mathbb{Z}$ and $\theta = 7\pi/6 + 2k\pi, 11\pi/6 + 2l\pi, k, l \in \mathbb{Z}$. Notice there is overlap at $\theta = \pi/2 + 2k\pi$ so we must take a limit to verify the slope. We see that

$$\lim_{\theta\to\pi/2} \frac{\cos\theta - \sin(2\theta)}{-\sin\theta - \cos(2\theta)} = \lim_{\theta\to\pi/2} \frac{-\sin\theta - 2\cos(2\theta)}{-\cos\theta + 2\sin(2\theta)} = \frac{1}{0}$$

so the horizontal tangents only occur at $\theta = \pi/2 + (2k+1)\pi, k \in \mathbb{Z}$ and $\theta = \pi/6 + 2k\pi, 5\pi/6 + 2l\pi, k, l \in \mathbb{Z}$.

(c) $dy/d\theta = \sec\theta\tan\theta\sin\theta + \tan\theta$ and $dx/d\theta = \tan\theta - \sin\theta\sec\theta = 0$. The slope $\frac{dy}{dx}$ is undefined. So for all θ we have a vertical tangent and no horizontal tangents.

(d) $dy/d\theta = e^{\theta}(\sin\theta + \cos\theta)$ and $dx/d\theta = e^{\theta}(\cos\theta - \sin\theta)$ so the slope is $\frac{dy}{dx} = \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$. The horizontal tangents are where $\tan\theta = -1$ which occurs at $3\pi/4 + k\pi, k \in \mathbb{Z}$ and the vertical tangents are where $\tan\theta = 1$ which occurs at $\pi/4 + k\pi, k \in \mathbb{Z}$.

2 Computing Areas

(a) Find the area of the region that lies inside $r = 3\cos\theta$ and outside $r = 1 + \cos\theta$.

(b) Find the area of the region that lies inside both curves $r = \sin 2\theta$ and $r = \cos 2\theta$.

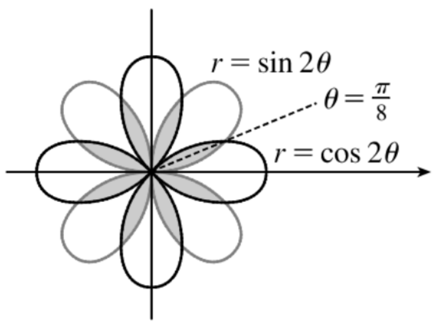
Solution:

(a) First we find where these two curves intersect. Notice that if $3\cos\theta = 1 + \cos\theta \Leftrightarrow \cos\theta = 1/2$ so $\theta = -\pi/3, \pi/3$. Then by symmetry, the area is

$$2\int_0^{\pi/3} \frac{1}{2}[(3\cos\theta)^2 - (1+\cos\theta)^2]d\theta = \int_0^{\pi/3} 8\cos^2\theta - 2\cos\theta - 1$$
$$= \int_0^{\pi/3} 3 + 4\cos 2\theta - 2\cos\theta d\theta$$
$$= \pi.$$

(b) Again we first see where they intersect. $\sin 2\theta = \cos 2\theta \Rightarrow \tan 2\theta = 1 \Rightarrow 2\theta = \pi/4 \Rightarrow \theta = \pi/8$. Notice we have 16 such regions are indicated in the picture. Hence the area is

$$16\int_0^{\pi/8} \frac{1}{2}\sin^2 2\theta d\theta = 4\int_0^{\pi/8}(1-\cos 4\theta)d\theta = 4\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/8} = \pi/2 - 1.$$



3 Computing Arc Lengths

Using the appropriate formula, find the length of the curve.

(a) $r = 2\cos\theta$, $0 \leq \theta \leq \pi$.

(b) $r = \theta^2$, $0 \leq \theta \leq 2\pi$.

Solution:

(a) Applying the polar formula we have

$$L = \int_0^{\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta} d\theta = 2\pi.$$

(b) Applying the polar formula we have

$$L = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2} d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2 + 4} d\theta = \frac{1}{3}(\theta^2 + 4)^{3/2}\Big|_0^{2\pi} = \frac{8}{3}[(\pi^2 + 1)^{3/2} - 1].$$

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.