Discussion 20 Worksheet Answers
More Stokes and Divergence
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MATH 53 Multivariable Calculus

1 Archimedes' Principle

The Force exerted on a solid D with surface S fully submerged in water is given by

$$\mathbf{F} = \iint_S -p\hat{\mathbf{n}}\, dS$$

where the pressure p is given by $-\rho g z$ (ρ is the density of water and g the gravitational acceleration) if we assume that the surface of the water is at $z = 0$. Use the divergence theorem to show Archimedes' principle $\vec{F} = \rho g \operatorname{vol}(D)$.
Hint: Compute $\mathbf{F}\cdot\mathbf{i}, \mathbf{F}\cdot\mathbf{j}, \mathbf{F}\cdot\mathbf{k}$

2 identities

Prove each of these identities, assuming that D is a solid region in 3D space and $S = \partial D$.

(i) $\iint_S \mathbf{a}\cdot\mathbf{n}\, dS = 0$ where $\mathbf{a}$ is any constant vector.

Solution: This follows from the divergence theorem because $\nabla\cdot\mathbf{a} = 0$.

(ii) $\mathrm{Vol}(D) = \frac{1}{3}\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = (x, y, z)$

Solution: Also follows from the divergence theorem because $\nabla\cdot\mathbf{F} = 3$.

(iii) $\iint_S (f\nabla g)\cdot d\mathbf{S} = \iiint_D f\nabla^2 g + \nabla f\cdot\nabla g\, dV$

Solution: Since $\nabla\cdot(f\nabla g) = \nabla f\cdot\nabla g + f\nabla^2 g$ so the divergence theorem fives us this identity.

3 parametrizing surfaces

Parametrize the following surfaces

(a) $x^2 + y^2 + 1 = z^2$ *(Hint: graph)*

(b) $x^2 + y^2 = z^2$ *(try spherical coordinates)*

(c) $x^2 + y^2 = (1 + z^2)^2$ *(Hint: cylindrical coordinates)*

(d) $y^2 + z^2 = e^x$

(e) $e^x = 1 + y^2 + 2\cos^2 z$

(f) $(x^2 + y^2 + z^2)^{3/2} = 2x^2 + 2y^2 + z^2$

(g) $x\sin z = y\cos z$

4 Stokes / surface integrals

1. Compute $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = (x^2, 2z, -3y)$ and S is the portion of $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$.

2. Compute $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ where $\vec{F} = (y, -x, yx^3)$ and S is the portion of the sphere of radius 4 with $z \geq 0$ and the upwards orientation.

3. Compute $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = (\sin(\pi x), zy^3, z^2 + 4x)$ where S is the surface of the box $-1 \leq x \leq 2$, $0 \leq y \leq 1$, and $1 \leq z \leq 4$, oriented outwards.

Solution:

1. Parametrize the surface by $x = x$, $y = 2\cos\theta$, and $z = 2\sin\theta$ for $0 \leq \theta \leq 2\pi$, $0 \leq x \leq 3 - 2\cos\theta$. Then $\vec{r}_x = (1, 0, 0)$ and $\vec{r}_\theta = (0, 2\cos\theta, -2\sin\theta)$. So $\vec{r}_x \times \vec{r}_\theta = -2\sin\theta\vec{j} - 2\cos\theta\vec{k}$. Our integral then becomes

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^{3-2\cos\theta} (0, 2\cos\theta, -2\sin\theta) \cdot (x^2, 4\cos\theta, -6\sin\theta) dx d\theta \\ &= \int_0^{2\pi} \int_0^{3-2\cos\theta} 4\sin\theta\cos\theta dx d\theta \\ &= \int_0^{2\pi} 12\sin\theta\cos\theta - 8\sin\theta\cos^2\theta d\theta = 0.\end{aligned}$$

2. We use Stokes' theorem and then Green's theorem. Note that the boundary circle C is the circle of radius 4 centered at the origin in the xy-plane. Let D be the disk of radius 4 enclosed by C in the xy-plane. Then

$$\begin{aligned}\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_C y dx - x dy \\ &= \iint_D -2 dA = -2 \cdot (16\pi) = -32\pi.\end{aligned}$$

3. We use the divergence theorem. Note that $\nabla \cdot \vec{F} = \pi\cos(\pi x) + 3y^2 z + 2z$. So our integral becomes

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \int_{-1}^{2} \int_0^1 \int_1^4 (\pi\cos(\pi x) + 3y^2 z + 2z) dz dy dx \\ &= \int_{-1}^{2} \int_0^1 3\pi\cos(\pi x) + \frac{45}{2} y^2 + 15 dy dx \\ &= \int_{-1}^{2} 3\pi\cos(\pi x) + \frac{15}{2} + 15 dx = \frac{135}{2}.\end{aligned}$$

5 Stokes' theorem

1. Verify Stokes' theorem for the following surfaces S and vector fields $\mathbf{F}$.

(a) $\mathbf{F}(x, y, z) = (y, z, x)$, S is the hemisphere $x^2 + y^2 + z^2 = 1, y \geq 0$.

Solution: We assume that the hemisphere is oriented "outward". $\nabla \times \mathbf{F} = (-1, -1, -1)$ so using that the normal vector for spherical coordinates (here we are using the y-axis instead of the z-axis as we usually do) $\mathbf{r}(\phi, \theta)$ is $\sin\phi \mathbf{r}(\phi, \theta)$ we have

$$\begin{aligned}\iint_S (-1, -1, -1)\, d\mathbf{S} &= -\int_0^{\pi/2}\int_0^{2\pi} \sin^2\phi(\cos\theta + \sin\theta) + \sin\phi\cos\phi\, d\theta\, d\phi \\ &= -2\pi \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi \\ &= -\pi\end{aligned}$$

On the other hand, the boundary of this surface is parametrized by $\mathbf{r}(t) = (\cos t, 0, -\sin t)$ so

$$\begin{aligned}\int_{\partial S} \mathbf{F}\, d\mathbf{r} &= \int_0^{2\pi} (0, -\sin t, \cos t) \cdot (-\sin t, 0, -\cos t)\, dt \\ &= -\pi\end{aligned}$$

(b) $\mathbf{F}(x, y, z) = (-y, x, -2)$ and S is the cone $z^2 = x^2 + y^2, 0 \leq z \leq 4$.

Solution: First compute $\nabla \times \mathbf{F} = (0, 0, 2)$ This cone is describe in spherical coordinates by $\phi = \pi/4, \rho \leq 4$ so we can parametrize it as $\mathbf{r}(\rho, \theta) = \frac{1}{\sqrt{2}}(\rho\cos\theta, \rho\sin\theta, \rho)$. This gives $\mathbf{r}_\rho \times \mathbf{r}_\theta = \frac{\rho}{2}(-\cos\theta, \sin\theta, 1)$. This is oriented upwards so we have to flip the sign to get the desired downward orientation. Now we compute

$$\begin{aligned}\iint_S \mathbf{F}\, d\mathbf{S} &= \int_0^4 \int_0^{2\pi} (0, 0, 2) \cdot \frac{\rho}{2}(\cos\theta, -\sin\theta, -1)\, d\theta\, d\phi \\ &= -4\pi \int_0^4 \rho\, d\rho = -32\pi\end{aligned}$$

On the other hand, the boundary can be parametrized by $\mathbf{r}(t) = (4\cos t, -4\sin t, 4)$, yielding

$$\int_{\partial S} = \int_0^{2\pi} (4\sin t, 4\cos t, -2) \cdot (-4\sin t, -4\cos t, 0)\, dt = -32\pi$$

6 Past final problems

1. Let C be the spiral $r = \theta$ between $\theta = 0$ and $\theta = a$, for some $a > 0$.

   a) Set up an integral to find the integral of xy over C with respect to arc length. Do not attempt to evaluate the integral.

b) Write down a vector field $\vec{F}$ (not depending on a) such that $\int_C \vec{F} \cdot d\vec{r}$ is equal to the integral in (a).

2. Calculate $\iint_S \vec{F} \cdot d\vec{S}$ where S is the unit sphere $x^2 + y^2 + z^2 = 1$, oriented using the outward pointing normal, and

$$\vec{F} = (x + \sin y, y + \sin z, z + \sin x).$$

3. Let $\vec{r}_1$ and $\vec{r}_2$ be two parametric curves in three dimensions that satisfy

$$\frac{d\vec{r}_1}{dt} = \vec{r}_2 - \vec{r}_1$$
$$\frac{d\vec{r}_2}{dt} = \vec{r}_2 + \vec{r}_1.$$

Show that $\vec{r}_1 \times \vec{r}_2$ is constant in time.

4. Find the volume of the solid enclosed by the surface

$$(x^2 + y^2 + z^2)^2 = 2z(x^2 + y^2).$$

Solution:

1. a) We can parametrize the curve by $x = \theta\cos\theta$, $y = \theta\sin\theta$. Then

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2} = \sqrt{1+\theta^2}.$$

So our arc length integral is

$$\int_0^a xy\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int_0^a \theta^2\cos\theta\sin\theta\sqrt{1+\theta^2}\,d\theta.$$

b) The easiest way to do this is to force the first component of $\vec{F}$ to be zero. If F_2 is the second component of $\vec{F}$, we want

$$\int_0^a \theta^2\cos\theta\sin\theta\sqrt{1+\theta^2}\,d\theta = \int_0^a \vec{F}(\vec{r}(\theta))\cdot\vec{r}'(\theta)\,d\theta = \int_0^a F_2(\vec{r}(\theta))y'(\theta)\,d\theta.$$

So we can set

$$F_2(\vec{r}(\theta)) = \frac{\theta^2\cos\theta\sin\theta\sqrt{1+\theta^2}}{y'(\theta)} = \frac{\theta^2\cos\theta\sin\theta\sqrt{1+\theta^2}}{\sin\theta + \theta\cos\theta}.$$

Rewriting this in terms of the variables x and y, we get

$$\vec{F} = \frac{xy\sqrt{1+x^2+y^2}\sqrt{x^2+y^2}}{y + x\sqrt{x^2+y^2}}\vec{j}.$$

Plenty of other vector fields would also work; this is just the easiest one to find.

2. Note $\nabla\cdot\vec{F} = 3$, so by the divergence theorem, the result is just 3 times the volume of the unit ball. Thus the answer is 12π.

3. To simplify notation we replace d/dt with primes. The product rule still holds for cross products (prove it!), so

$$\begin{aligned}(\vec{r}_1\times\vec{r}_2)' &= \vec{r}_1{}'\times\vec{r}_2 + \vec{r}_1\times\vec{r}_2{}'\\ &= (\vec{r}_2 - \vec{r}_1)\times\vec{r}_2 + \vec{r}_1\times(\vec{r}_2+\vec{r}_1)\\ &= -\vec{r}_1\times\vec{r}_2 + \vec{r}_1\times\vec{r}_2 = 0.\end{aligned}$$

Thus the time derivative of $\vec{r}_1\times\vec{r}_2$ is zero; this is the same as saying that $\vec{r}_1\times\vec{r}_2$ is constant in time.

4. We rewrite this surface in spherical coordinates as $\rho^4 = 2\rho^3\cos\phi\sin^2\phi$. Now we find the bounds for an integral in spherical coordinates. Note that θ is unconstrained, so $0\le\theta\le 2\pi$. We see that $\cos\phi$ must be positive, so $0\le\phi\le\pi/2$. Our defining equation shows $0\le\rho\le 2\cos\phi\sin\phi$. So our volume integral is

$$2\pi\int_0^{\pi/2}\int_0^{2\cos\phi\sin^2\phi}\rho^2\sin\phi\,d\rho\,d\phi = \frac{16\pi}{3}\int_0^{\pi/2}\cos^3\phi\sin^7\phi\,d\phi = \frac{2\pi}{15}.$$