# Discussion 2 Worksheet Answers

## Tangents, Area, Arclength

Date: 8/30/2021

MATH 53 Multivariable Calculus

## 1 Computing Tangents

Compute the slopes of the following curves at a point in time $t$. Find the points where the tangents are vertical and horizontal and compute the second derivative $d^2y/dx^2$ at the horizontal points.

(a) $x = \cos t, y = \sin t$

(b) $x = t^2 - 1 \; y = t^3 - t$

(c) $x = e^t - 1, y = \sin t$

(d) $x = e^t - t, y = \cos t$

**Solution:**

(a) $x'(t) = -\sin t, y'(t) = \cos t$ so the slope is $\frac{dy}{dx} = -\frac{\cos t}{\sin t} = -\cot t$. The tangents are horizontal where $x'(t) = 0$, i.e. at $t = \pi/2 + k\pi$ and vertical where $y'(t) = 0$, i.e. at $k\pi, k \in \mathbb{Z}$. For the second derivative, we get

$$\frac{d^2y}{dx^2} = \frac{1}{x'(t)} \frac{d}{dt} \frac{dy}{dx} = \frac{-1}{\sin t} \left( \frac{1}{\tan^2 t \cos^2 t} \right) = \frac{-1}{\sin^3 t}$$

which is $-1$ at $t = \pi/2 + k\pi$ for even $k$ and $+1$ for odd $k$.

(b) $x'(t) = 2t, y'(t) = 3t^2 - 1$ so the slope is $\frac{dy}{dx} = \frac{3t^2-1}{2t}$. The tangents are horizontal at $t = \pm 1/\sqrt{3}$ and vertical at $t = 0$. Note that the point $(0, 0)$ which corresponds to $t = \pm 1$ has two different tangents. The second derivative is

$$\frac{d^2y}{dx^2} = \frac{1}{2t} \frac{d}{dt} \frac{3t^2 - 1}{2t} = \frac{12t^2 - 2(3t^2 - 1)}{8t^3} = \frac{6t^2 + 2}{8t^3}$$

So at the horizontal points we get $\pm 3\sqrt{3}/2$.

(c) $x'(t) = e^t, y'(t) = \cos t$ so the slope is $\frac{dy}{dx} = e^{-t} \cos t$. The tangents are horizontal at $t = \pi/2 + k\pi, k \in \mathbb{Z}$ and never vertical. The second derivative is

$$\frac{d^2y}{dx^2} = e^{-t} \frac{d}{dt} e^{-t} \cos t = -e^{-2t}(\sin t + \cos t)$$

At the points with horizontal tangents we get $(-1)^{k+1} e^{-\pi - 2\pi k}$.

(d) $x'(t) = e^t - 1$ and $y'(t) = -\sin(t)$. We have $x'(0) = y'(0) = 0$, so the slope as 0 is undefined (although L'Hôpital's rule makes it reasonable to assign slope -1 to this point). Horizontal tangents occur at $t = k\pi$ where $k \in \mathbb{Z}$ but $k \neq 0$. The second derivative is (after some simplification)

$$\frac{d^2y}{dx^2} = \frac{\cos t + e^t(\sin t - \cos t)}{(e^t - 1)^3}$$

so at $t = k\pi$ it's $(-1)^{k+1} \left(1 - e^{k\pi}\right)^{-2}$.

## 2 Computing Areas

Using the appropriate formula, find the area in question.

(a) Use the parametric equations of an ellipse, $x = a \cos \theta$, $y = b \sin \theta$, $0 \leq \theta \leq 2\pi$, to find the area that it encloses.

(b) Find the area enclosed by the $x$-axis and the curve $x = t^3 + 1$, $y = 2t - t^2$.

(c) Find the area of the region enclosed by the astroid $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

**Solution:**

(a) Applying the formula and by symmetry we have

$$4\int_0^a y\,dx = 4\int_{\pi/2}^0 b\sin\theta(-a\sin\theta)d\theta = 2ab\int_0^{\pi/2} 1-\cos 2\theta d\theta = \pi ab$$

as the area.

(b) Notice that $y = 2t - t^2$ intersects that $x$-axis at $t = 0$ and $t = 2$. The corresponding values of $x$ are 1 and 9 so the area in question is

$$\int_1^9 y\,dx = \int_0^2 (2t-t^2)(3t^2)dt = 3\left[\frac{t^4}{2} - \frac{t^5}{5}\right]_0^2 = \frac{24}{5}.$$

(c) Applying the formula and by symmetry we have

$$4\int_0^a y\,dx = 4\int_{\pi/2}^0 a\sin^3\theta(-3a\cos^2\theta\sin\theta)d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta d\theta = \frac{3\pi a^2}{8}$$

as the area.

# 3 Computing Arc Lengths

Using the appropriate formula, find the length of the curve.

(a) $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \le t \le 1$.

(b) $x = e^t - t$, $y = 4e^{t/2}$, $0 \le t \le 2$.

(c) $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le \pi$.

**Solution:**

(a) Applying the formula we have

$$L = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2} dt = \int_0^1 6t\sqrt{1+t^2} dt$$
$$= [2(1+t^2)^{3/2}]_0^1$$
$$= 4\sqrt{2} - 2.$$

(b) Applying the formula we have

$$L = \int_0^2 \sqrt{(e^t - 1)^2 + (2e^{t/2})^2} dt = \int_0^2 \sqrt{e^{2t} + 2e^t + 1} dt = \int_0^2 e^t + 1 dt = e^2 + 1.$$

(c) Applying the formula we have

$$L = \int_0^\pi \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \cos t + e^t \sin t)^2} dt = \int_0^\pi \sqrt{2e^{2t}\cos^2 t + 2e^{2t}\sin^2 t} dt$$
$$= \int_0^\pi \sqrt{2} e^t dt$$
$$= \sqrt{2}(e^\pi - 1).$$

## 4 True/False

(a) T F The parametric representation of a curve is unique.

**Solution:** False. We can transform $\theta \to 2\theta$ and recover the same curve

(b) T F When integrating, we can replace $\sin^2\theta$ with $(1 - \cos 2\theta)/2$.

**Solution:** True, this can be deduced from the general addition theorem $\cos 2\theta = 1 - 2\sin^2\theta$.

(c) T F A (parametric) curve can only be described in either Cartesian coordinates $x = f(t), y = g(t)$ or in polar coordinates $r = f(\theta)$, but not both.

**Solution:** False. Typically a curve can be described both ways.

(d) T F $\sin(2t) = 2\sin t \cos t$

**Solution:** True, this can be deduced from the general addition theorem $\sin(s+t) = \sin s \cos t + \cos s \sin t$.

**Note:** These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.