Discussion 19 Worksheet Answers

Surface integrals of vector fields

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MATH 53 Multivariable Calculus

1 Vector Surface Integrals

Compute the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}$$

for a given vector field $\vec{F}(x, y, z)$ over the surface S.

(a) $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$ over the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1, 0 \leq y \leq 1$.

Solution: This surface is the graph of $f(x, y) = 4 - x^2 - y^2$ so we have $\vec{N} = \langle -f_x, -f_y, 1 \rangle = \langle 2x, 2y, 1 \rangle$. Hence the integral is

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \int_0^1 \int_0^1 2x \cdot xy + 2y \cdot yz + zx \, dx \, dy \\
&= \int_0^1 \int_0^1 2x^2 y + (2y^2 + x)(4 - x^2 - y^2) \, dx \, dy \\
&= \frac{2}{6} + \frac{8}{3} - \frac{2}{9} - \frac{2}{5} + \frac{4}{2} - \frac{1}{4} - \frac{1}{6} = \frac{713}{180}
\end{aligned}$$

(b) $\vec{F}(x, y, z) = \langle x, y, z^2 \rangle$ and S is the unit sphere centered at the origin.

Solution: We use the spherical coordinate parametrization from problem **??**. We have

$$\begin{aligned}
\vec{F} &= \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos^2\phi \rangle \\
\vec{F} \cdot \vec{N} &= \sin\phi \left(\sin^2\phi \cos^2\theta + \sin^2\phi \sin^2\theta + \cos^3\phi \right) \\
&= \sin^3\phi + \sin\phi \cos^3\phi \\
\iint_S \vec{F} \cdot d\vec{S} &= \int_0^\pi \int_0^{2\pi} \sin^3\phi + \sin\phi \cos^3\phi \, d\theta \, d\phi \\
&= 2\pi \int_{-1}^1 (1 - u^2 + u^3) \, du = \frac{8\pi}{3}
\end{aligned}$$

2 Oriented Surfaces

Evaluate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ for the given vector field $\vec{F}$ and oriented surface S. For closed surfaces, use the positive (outward) orientation.

(a) $\vec{F}(x,y,z) = \langle ze^{xy}, -3ze^{xy}, xy\rangle$. S is the parallelogram $x = u+v, y = u-v, z = 1+2u+v, 0 \le u \le 2, 0 \le v \le 1$ oriented upwards.

Solution: Let $\vec{r}(u,v) = \langle u+v, u-v, 1+2u+v\rangle$. Then $\vec{r}_u \times \vec{r}_v = \langle 3, 1, -2\rangle$. We get $\vec{F}(\vec{r}(u,v)) = \langle (1+2u+v)e^{u^2-v^2}, -3(1+2u+v)e^{u^2-v^2}, u^2-v^2\rangle$. Since the z-component of $\vec{r}_u \times \vec{r}_v$ is negative, we use $-(\vec{r}_u \times \vec{r}_v)$. Thus,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (-(\vec{r}_u \times \vec{r}_v))dA = \int_0^1 \int_0^2 2(u^2 - v^2)dudv = 4.$$

(b) $\vec{F}(x,y,z) = \langle 0, y, -z\rangle$ and S consists of the paraboloid $y = x^2 + z^2, 0 \le y \le 1$, and the disk $x^2 + z^2 \le 1, y = 1$.

Solution: Let S_1 be the paraboloid and S_2 be the disk. Since S is closed, we use the outward orientation. On S_1 we have $\vec{F}(\vec{r}(x,z)) = \langle 0, x^2+z^2, -z\rangle$ and $\vec{r}_x \times \vec{r}_z = \langle 1, 2x, 0\rangle \times \langle 0, 2z, 1\rangle = \langle 2x, -1, 2z\rangle$. Then

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{x^2+z^2 \le 1} (-(x^2+z^2) - 2z^2)dA = -\int_0^{2\pi}\int_0^1 (r^2 + 2r^2 \sin^2\theta)rdrd\theta = -\pi$$

and on S_2 we have $\vec{F}(\vec{r}(x,z)) = \langle 0, 1, -z\rangle$ and $\vec{r}_z \times \vec{r}_x = \langle 0, 1, 0\rangle$. Then $\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{x^2+z^2 \le 1} dA = \pi$.

(c) Do (b) using the divergence theorem.

(d) $\vec{F}(x,y,z) = \langle x^2, y^2, z^2\rangle$ and S is the boundary of the solid half cylinder $0 \le z \le \sqrt{1-y^2}, 0 \le x \le 2$.

Solution: Here S has four surfaces. S_1 is the portion of the cylinder, S_2 is the bottom surfaces (lies on xy-plane), S_3 is the front half disk at $x = 2$ and S_4 is the back half disk at $x = 0$.
On S_1 we have $\vec{r}(x,y) = \langle x, y, \sqrt{1-y^2}\rangle$ so $\vec{r}_x = \langle 1, 0, 0\rangle$ and $\vec{r}_y = \langle 0, 1, -y(1-y^2)^{-1/2}$. Then

$$\iint_{S_1} F \cdot d\vec{S} = \int_0^2 \int_{-1}^1 \vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y)dxdy = \int_0^2 \int_{-1}^1 y^3(1-y^2)^{-1/2} + (1-y^2)dydx = 8/3.$$

On S_2 we have $z = 0$ with downward orientation so $\iint_{S_2} \vec{F} \cdot d\vec{S} = \int_0^2 \int_{-1}^1 -z^2 dydx = 0$.
On S_3, the surfaces is $x = 2$ for $-1 \le y \le 1$ and $0 \le z \le \sqrt{1-y^2}$ oriented in the positive x-direction. Hence, $\vec{r}_y \times r_z = \vec{i}$ so $\iint_{S_3} \vec{F} \cdot d\vec{S} = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x^2 dzdy = 4\int_{-1}^1 \int_0^{\sqrt{1-y^2}} dzdy = 2\pi$.
On S_4, the surfaces is $x = 0$ for $-1 \le y \le 1$ and $0 \le z \le \sqrt{1-y^2}$ oriented in the negative x-direction. Hence, $\vec{r}_z \times r_y = -\vec{i}$ so $\iint_{S_3} \vec{F} \cdot d\vec{S} = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x^2 dzdy = 0\int_{-1}^1 \int_0^{\sqrt{1-y^2}} dzdy = 0$.
Summing these we get, $2\pi + 8/3$.

(e) Do (d) using the divergence theorem.

3 Divergence theorem?

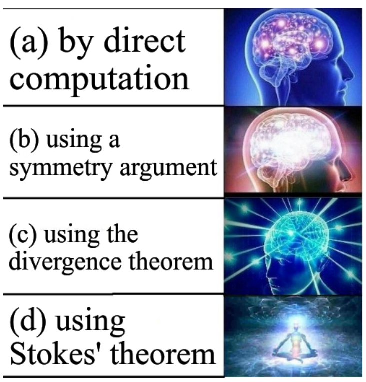
- Let S be the same cylinder $x^2 + y^2 = 1, -1 \leq z \leq 1$, plus its top and bottom caps. Compute the flux of the vector field

$$\vec{F}(x, y, z) = \begin{pmatrix} -\sin \pi y \\ -\cos \pi x \\ xy \end{pmatrix}$$

both directly and by using the divergence theorem.

> **Solution:** The vector field is incompressible, so by the divergence theorem we immediately know that the flux has to be zero.
> If we do a direct computation the sides of the cylinder will have zero contribution because $\vec{n} \perp \vec{F}$ there and the contribution from the top and bottom caps will cancel.

- Let $\vec{F}(x, y, z) = (x^2, yz, xz)$ and evaluate $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$, where S is the unit sphere...



4 Challenge

Let $\vec{F}(\vec{r}) = c\vec{r}/|\vec{r}|^3$ for some constant c and $\vec{r} = \langle x, y, z \rangle$. Show that the flux of $\vec{F}$ across a sphere S with center the origin is independent of the radius of S.

> **Solution:** Let R be the radius of the sphere centered at the origin. Then $|\vec{r}| = R$ and $\vec{F}(\vec{r}) = (c/R^3)\langle x, y, z \rangle$. We know we can parametrize the sphere S as $\vec{r}(\phi, \theta) = \langle R \sin \phi \cos \theta, R \sin \phi \sin \theta, R \cos \phi \rangle, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$. Then the outward orientation is given by $\vec{r}_\phi \times \vec{r}_\theta = \langle R^2 \sin^2 \phi \cos \theta, R^2 \sin^2 \phi \sin \theta, R^2 \sin \phi \cos \phi \rangle$. The flux of $\vec{F}$ across S is
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> $$\iint_S \vec{F} \cdot d\vec{S} = \int_0^\pi \int_0^{2\pi} \frac{c}{R^3} \cdot R^3 (\sin^3 \phi + \sin \phi \cos^2 \phi) d\theta d\phi = 4\pi c,$$
>
> which is independent of R.

5 True/False

(a) T F Consider a sphere lying inside the smallest possible cylinder that can contain it. Then the ratio of the volume of the sphere to the volume of the cylinder is the same as the ratio of the surface areas.

Solution: **True.** If the sphere has radius R then the cylinder will have radius R and height $2R$ so it will have volume $\pi R^2 \cdot 2R$ and surface area $2 \cdot \pi R^2 + 2\pi R \cdot 2R = 6\pi R^2$. Comparing with the volume and area formulas for spheres we see that the ratio is $\frac{2}{3}$ in both cases.

(b) T F If we interchange the variables u, v in a parametrization of a surface (e.g. $\vec{r}(u,v) = \langle u^2, uv, v^2 - u\rangle$ becomes $\vec{r'}(u,v) = \langle v^2, uv, u^2 - v\rangle$) the surface this describes stays the same.

Solution: **True** because the range of the functions $\vec{r}$ and $\vec{r'}$ will be the same.

(c) T F If we interchange the variables u, v as above and use $\vec{r'}$ to compute a vector surface integral, we will get the same result as if using $\vec{r}$.

Solution: **False.** $\vec{r}_u = \vec{r'}_v$ and $\vec{r}_v = \vec{r'}_u$, so $\vec{r}_u \times \vec{r}_v = -\vec{r}_v \times \vec{r}_u = -\vec{r'}_u \times \vec{r'}_v$. Therefore the integral computed with $\vec{r'}$ will have opposite sign.

(d) T F Consider the **unit** normal vector of a parametrized surface (which can be obtained as $\vec{N}(u,v) = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$). The u- and v-derivatives of $\vec{N}$ are always parallel to the surface.

Solution: **True.** We know that $1 = |\vec{N}|^2 = \vec{N} \cdot \vec{N}$ and taking the u-derivative of this gives $0 = 2\vec{N} \cdot \vec{N}_u$ so $\vec{N}_u$ is perpendicular to $\vec{N}$. But since there is only one direction that is perpendicular to the surface, $\vec{N}_u$ must be parallel to the surface. The same reasoning applies to $\vec{N}_v$.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.