# Discussion 18 Worksheet Answers
## Parametric surfaces and surface integrals

Date: 11/10/2021

MATH 53 Multivariable Calculus

## 1 Sphere Parametrization

Consider a sphere of radius $R$ centered at the origin. We know that the sphere can be parametrized by

$$\vec{r}(\phi, \theta) = \begin{pmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{pmatrix},$$

$0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$.

(a) Compute the partial derivatives of $\vec{r}(\phi, \theta)$.

**Solution:** $\vec{r}_\phi = \langle R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\phi\rangle$ and $\vec{r}_\theta = \langle -R\sin\phi\sin\theta, R\sin\phi\cos\theta, 0\rangle$

(b) Compute the normal vector $\vec{r}_u \times \vec{r}_v$ produced by this parametrization. Express it in terms of $\phi, \theta$ and $x, y, z$.

**Solution:** $\vec{N}(\phi, \theta) = \vec{r}_\phi \times \vec{r}_\theta = \langle R^2\sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2\sin\phi\cos\phi\rangle = R\sin\phi\langle R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi\rangle = R\sin\phi\,\vec{r}(\theta, \phi) = R\sin\phi \cdot \langle x, y, z\rangle$

(c) Use the magnitude of the normal vector (the "Jacobian") to compute the area of the unit sphere.

**Solution:** We can now obtain the area of the sphere by integrating $|\vec{N}| = R^2\sin\phi$ over $0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$:

$$A = \int_0^\pi \int_0^{2\pi} R^2\sin\phi\, d\theta\, d\phi = 2\pi R^2 \int_0^\pi \sin\phi\, d\phi = 4\pi R^2$$

(d) Compute the surface integral of $z^2$ over the sphere.

**Solution:** This is

$$A = \int_0^\pi \int_0^{2\pi} R^2\cos\phi R^2\sin\phi\phi\phi\, d\theta\, d\phi = 2\pi R^4 \int_0^\pi \cos^2\phi\sin\phi\, d\phi = 4\pi R^4/3$$

## 2 Surface Areas

Parametrize the following surfaces in an appropriate way (if they are not already parametrized) and compute their normal vectors and area.

(a) The portion of the elliptic paraboloid $z = x^2 + y^2$ lying over the unit disk.

**Solution:** This surface is the graph of $f(x, y) = x^2 + y^2$, so we know that

$$\vec{N} = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \langle -f_x, -f_y, 1 \rangle = \langle -2x, -2y, 1 \rangle.$$

The area is computed by the following integral over the unit disk $D$, which we compute in polar coordinates and using the substitution $u = 1 + 4r^2$

$$\begin{aligned} \int_D |\vec{N}|\, dA = \int_D \sqrt{1 + 4x^2 + 4y^2}\, dA &= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\, r\, dr\, d\phi \\ &= 2\pi \int_1^5 \sqrt{u} \frac{1}{8}\, du = \frac{\pi}{6} \left( 5^{3/2} - 1 \right) \end{aligned}$$

(b) The ellipsoid $2z^2 + x^2 + y^2 = 1$. You don't need to evaluate the integral, but you can do it using

$$\int \sqrt{1 + x^2}\, dx = \frac{1}{2} x \sqrt{1 + x^2} + \frac{1}{2} \ln \left( x + \sqrt{1 + x^2} \right) + C.$$

**Solution:** This ellipsoid can be obtained by taking the unit sphere and squishing the $z$-coordinate by a factor of $\frac{1}{2}$. Hence we can parametrize it by (cf problem 1)

$$\vec{r}(\phi, \theta) = \left\langle \sin\phi \cos\theta, \sin\phi \sin\theta, \frac{1}{2} \cos\phi \right\rangle$$

Now we obtain

$$\begin{aligned} \vec{r}_\phi &= \left\langle \cos\phi \cos\theta, \cos\phi \sin\theta, -\frac{1}{2} \sin\phi \right\rangle \\ \vec{r}_\theta &= \langle -\sin\phi \sin\theta, \sin\phi \cos\theta, 0 \rangle \\ \vec{N} &= \vec{r}_\phi \times \vec{r}_\theta = \left\langle \frac{1}{2} \sin^2\phi \cos\theta, \frac{1}{2} \sin^2\phi \sin\theta, \sin\phi \cos\phi \right\rangle \end{aligned}$$

and we can set up an integral for the area:

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^\pi |\vec{N}|\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^\pi \sin\phi \sqrt{\frac{1}{4} \sin^2\phi \cos^2\theta + \frac{1}{4} \sin^2\phi \sin^2\theta + \cos^2\phi\, d\phi\, d\theta} \\ &= 2\pi \int_0^\pi \sin\phi \sqrt{\frac{1}{4} \sin^2\phi + \cos^2\phi\, d\phi} = \pi \int_0^\pi \sin\phi \sqrt{1 + 3\cos^2\phi}\, d\phi \\ &= \pi \int_{-1}^1 \sqrt{1 + 3u^2}\, du = \pi \frac{1}{\sqrt{3}} \frac{1}{2} \left[ x\sqrt{1 + x^2} + \ln \left( x + \sqrt{1 + x^2} \right) \right]_{x=-\sqrt{3}}^{x=\sqrt{3}} \\ &= \pi \left( 2 + \frac{\ln(2 + \sqrt{3})}{\sqrt{3}} \right) \approx 8.672 \end{aligned}$$

(c) The parametric surface $\vec{r}(u, v) = (u^2, uv, v^2/2)$ where $0 \le u \le 1, 0 \le v \le 2$.

**Solution:** $\vec{r}_u = \langle 2u, v, 0\rangle, \vec{r}_v = \langle 0, u, v\rangle$ so $\vec{N} = \vec{r}_u \times \vec{r}_v = \langle v^2, -2uv, 2u^2\rangle$. The area of the surface is

$$A = \iint_{[0,1]\times[0,2]} |\vec{N}|\, dA = \int_0^1 \int_0^2 \sqrt{v^4 + 4u^2v^2 + 4u^4}\, dv\, du = \int_0^1 \int_0^2 \sqrt{(v^2+2u^2)^2}\, dv\, du$$
$$= \int_0^1 \int_0^2 v^2 + 2u^2\, dv\, du = \frac{8}{3} + \frac{4}{3} = 4.$$

(d) The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

**Solution:** This is the graph of $f(x, y) = xy$, so

$$\vec{N} = \langle -f_x, -f_y, 1\rangle = \langle y, x, 1\rangle.$$

Restricting the surface to the part inside the cylinder corresponds to restricting the domain of $f$ to the unit disk $D$. The area of the surface is given by

$$A = \iint_D \sqrt{1 + x^2 + y^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2+1}r\, dr\, d\theta = \frac{2\pi}{3}\left(2\sqrt{2} - 1\right)$$

(The computation of the integral is analogous to problem 1.1).

# 3 Scalar Surface Integrals

Compute the surface integral

$$\iint_S f(x, y, z)\, dS$$

for the given function $f(x, y, z)$ over the surface $S$.

(a) $f(x, y, z) = x$ where $S$ is the surface $y = x^2 + 4z, 0 \leq x \leq 1, 0 \leq z \leq 1$.

**Solution:** Using $x$ and $z$ as parameters we have $\vec{r}(x, z) = \langle x, x^2 + 4z, z\rangle, 0 \leq x \leq 1, 0 \leq z \leq 1$. Then

$$\vec{N}(x, z) = \vec{r}_x \times \vec{r}_z = (\vec{i} + 2x\vec{j}) \times (4\vec{j} + \vec{k}) = \langle 2x, -1, 4\rangle$$
$$\iint_S x\, dS = \int_0^1 \int_0^1 x \cdot |\vec{N}(x, z)|\, dx\, dz = \int_0^1 x\sqrt{4x^2 + 17}$$
$$= \left[\frac{1}{8} \cdot \frac{2}{3}(4x^2 + 17)^{3/2}\right]_0^1 = \frac{7}{4}\sqrt{21} - \frac{17}{12}\sqrt{17}.$$

(b) $f(x, y, z) = (x^2 + y^2)z$ and $S$ the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$.

**Solution:** We parametrize the sphere as in problem 1, so $(x^2+y^2)z = R^3 \sin^2\phi\cos\phi = 8\sin^2\phi\cos\phi$ because $R = 2$. We compute

$$\begin{aligned}\iint_S f(x,y,z)\,dS &= \int_0^{\pi/2}\int_0^{2\pi} 8\sin^2\phi\cos\phi\cdot 4\sin\phi\,d\theta\,d\phi \\ &= 64\pi\int_0^{\pi/2}\sin^3\phi\cos\phi\,d\phi = 64\pi\int_0^1 u^3\,du = 16\pi\end{aligned}$$

**Note:** These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.