Discussion 17 Worksheet Answers

Some past exam problems

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MATH 53 Multivariable Calculus

1 Computing Curl and Divergence

For each of the following vector fields $\vec{F}$, compute its curl and divergence. State whether each vector field is irrotational, incompressible, or neither.

1. $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

Solution: We have

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 1 + 1 + 1 = 3$$

and

$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right)\vec{i} + \left(\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z\right)\vec{j} + \left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right)\vec{k} = \vec{0}.$$

Because $\nabla \times \vec{F} = \vec{0}$, we see that $\vec{F}$ is irrotational.

2. $\vec{F} = \langle y^2, z^3, x^4 \rangle$

Solution: We have

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}y^2 + \frac{\partial}{\partial y}z^3 + \frac{\partial}{\partial z}x^3 = 0 + 0 + 0 = 0$$

and

$$\begin{aligned}\nabla \times \vec{F} &= \left(\frac{\partial}{\partial y}x^4 - \frac{\partial}{\partial z}z^3\right)\vec{i} + \left(\frac{\partial}{\partial z}y^2 - \frac{\partial}{\partial x}x^4\right)\vec{j} + \left(\frac{\partial}{\partial x}z^3 - \frac{\partial}{\partial y}y^2\right)\vec{k} \\ &= -3z^2\vec{i} - 4x^3\vec{j} - 2y\vec{k}.\end{aligned}$$

Because $\nabla \cdot \vec{F} = 0$, we see that $\vec{F}$ is incompressible.

3. $\vec{F} = \langle y^2x, e^z, z^2 \rangle$

Solution: We have

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} y^2 x + \frac{\partial}{\partial y} e^z + \frac{\partial}{\partial z} z^2 = y^2 + 2z$$

and

$$\begin{aligned}\nabla \times \vec{F} &= \left(\frac{\partial}{\partial y} z^2 - \frac{\partial}{\partial z} e^z\right)\vec{i} + \left(\frac{\partial}{\partial z} y^2 x - \frac{\partial}{\partial x} z^2\right)\vec{j} + \left(\frac{\partial}{\partial x} e^z - \frac{\partial}{\partial y} y^2 x\right)\vec{k} \\ &= -e^z\vec{i} - 2yx\vec{k}.\end{aligned}$$

This vector field is neither irrotational nor incompressible (as $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ are both nonzero).

4. $\vec{F} = \nabla f$, where $f(x, y, z) = 2xye^{yz}$

Solution: We have

$$\nabla \cdot \vec{F} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 + (4xze^{yz} + 2xyz^2 e^{yz}) + 2xy^3 e^{yz} = (4xz + 2xyz^2 + 2xy^3)e^{yz}$$

and

$$\nabla \times \vec{F} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial y}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k} = \vec{0}$$

by Clairaut's theorem.

2 Divergence and Curl Identities

Let $\vec{F} = \langle P, Q, R\rangle$ and $\vec{G} = \langle P', Q', R'\rangle$ be vector fields on $\mathbb{R}^3$, and let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be functions on $\mathbb{R}^3$. Assume all of these are infinitely differentiable. Prove each of the following vector identities.

1. $\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G}$

Solution:

$$\begin{aligned}\nabla \cdot (\vec{F} + \vec{G}) &= (P + P')_x + (Q + Q')_y + (R + R')_z \\ &= P_x + Q_y + R_z + P'_x + Q'_y + R'_z \\ &= \nabla \cdot \vec{F} + \nabla \cdot \vec{G}.\end{aligned}$$

2. $\nabla \times (\vec{F} + \vec{G}) = \nabla \times \vec{F} + \nabla \times \vec{G}$

Solution: It's easiest to verify this component-by component. For the $\vec{i}$ component, we have

$$\begin{aligned}(\nabla \times (\vec{F}+\vec{G}))\cdot\vec{i} &= (R+R')_y - (Q+Q')_z \\ &= (R_y - Q_z) + (R'_y - Q'_z) \\ &= (\nabla\times\vec{F})\cdot\vec{i} + (\nabla\times\vec{G})\cdot\vec{i} \\ &= (\nabla\times\vec{F}+\nabla\times\vec{G})\cdot\vec{i}.\end{aligned}$$

Similarly we see that the desired equation holds when we look at the $\vec{j}$ and $\vec{k}$ components, proving the result.

3. $\nabla\cdot(f\vec{F}) = f(\nabla\cdot\vec{F}) + \vec{F}\cdot(\nabla f)$

Solution:

$$\begin{aligned}\nabla\cdot(f\vec{F}) &= (fP)_x + (fQ)_y + (fR)_z \\ &= (f_xP + fP_x) + (f_yQ + fQ_y) + (f_zR + fR_z) \\ &= (fP_x + fQ_y + fR_z) + (f_xP + f_yQ + f_zR) \\ &= f(\nabla\cdot\vec{F}) + \vec{F}\cdot(\nabla f).\end{aligned}$$

4. $\nabla\times(f\vec{F}) = f(\nabla\times\vec{F}) + (\nabla f)\times\vec{F}$

Solution: It's easiest to verify this component-by component. For the $\vec{i}$ component, we have

$$\begin{aligned}(\nabla\times(f\vec{F}))\cdot\vec{i} &= (fR)_y - (fQ)_z \\ &= f_yR + fR_y - f_zQ - fQ_z \\ &= f(R_y - Q_z) + (f_yR - f_zQ) \\ &= f(\nabla\times\vec{F})\cdot\vec{i} + ((\nabla f)\times\vec{F})\cdot\vec{i} \\ &= (f(\nabla\times\vec{F}) + (\nabla f)\times\vec{F})\cdot\vec{i}.\end{aligned}$$

A similar computation shows that the identity holds for the $\vec{j}$ and $\vec{k}$ components, proving the the identity.

5. $\nabla\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot(\nabla\times\vec{F}) - \vec{F}\cdot(\nabla\times\vec{G})$

Solution:

$$\begin{aligned}\nabla\cdot(\vec{F}\times\vec{G}) &= (QR' - RQ')_x + (RP' - PR')_y + (PQ' - QP')_z \\ &= Q_xR' + QR'_x - R_xQ' - RQ'_x + R_yP' + RP'_y - P_yR' - PR'_y + P_zQ' + PQ'_z - Q_zP' - QP'_z \\ &= P'(R_y - Q_z) + Q'(P_z - R_x) + R'(Q_x - P_y) - P(R'_y - Q'_z) - Q(P'_z - R'_x) - R(Q'_x - P'_y) \\ &= \vec{G}\cdot(\nabla\times\vec{F}) - \vec{F}\cdot(\nabla\times\vec{G}).\end{aligned}$$

6. $\nabla\cdot(\nabla f\times\nabla g) = 0$

Solution: This follows from the previous identity (taking $\vec{F} = \nabla f$ and $\vec{G} = \nabla g$) and the fact that $\nabla \times (\nabla f) = \nabla \times (\nabla g) = 0$.

7. $\nabla \times (\nabla \times \vec{F})) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$

Solution: We again work component-by-component. Note

$$\nabla \times \vec{F} = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k},$$

so

$$\begin{aligned}
\big(\nabla \times (\nabla \times \vec{F})\big) \cdot \vec{i} &= (Q_x - P_y)_y - (P_z - R_x)_z \\
&= Q_{xy} - P_{yy} - P_{zz} + R_{xz} \\
&= (P_{xx} + Q_{xy} + R_{xz}) - (P_{xx} + P_{yy} + P_{zz}) \\
&= (P_x + Q_y + R_z)_x - (P_{xx} + P_{yy} + P_{zz}) \\
&= (\nabla(\nabla \cdot \vec{F})) \cdot \vec{i} - (\nabla^2 \vec{F}) \cdot \vec{i} \\
&= \big(\nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}\big) \cdot \vec{i}.
\end{aligned}$$

Similar arguments show the identity also holds in the $\vec{j}$ and $\vec{k}$ components, proving the identity.

3 Challenge

Suppose you are given a pair of (infinitely differentiable) vector fields $\vec{E}$ and $\vec{B}$ in $\mathbb{R}^3$ in $\mathbb{R}^3$, and consider each vector field as additionally varying with respect to a variable t (in addition to the variables x, y, and z for $\mathbb{R}^3$). Suppose furthermore that these vector fields satisfy the "Maxwell equations in vacuum:"

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

for some constant $c^2 > 0$. Prove that these vector fields satisfy the "wave equations"

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}.$$

Here $\nabla^2 \vec{E}$ is the vector Laplacian

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2},$$

and $\nabla^2 \vec{B}$ is defined similarly (with $\vec{E}$ replaced by $\vec{B}$).

By completing this exercise, you are showing that the fundamental laws of electrodynamics suggest the possibility of electromagnetic waves, i.e. light. *Fiat lux!*

Solution: Problem 7 from Section 2 above shows that

$$\nabla^2 \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla \times (\nabla \times \vec{E}).$$

Substituting in the equations for $\nabla \cdot \vec{E}$ and $\nabla \times \vec{E}$ here gives

$$\nabla^2 \vec{E} = 0 - \nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = \nabla \times \frac{\partial \vec{B}}{\partial t}.$$

By Clairaut's theorem, we can replace this with

$$\nabla^2 \vec{E} = \frac{\partial}{\partial t} \left(\nabla \times \vec{B} \right).$$

Substituting in the equation for $\nabla \times \vec{B}$ gives

$$\nabla^2 \vec{E} = \frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right) = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}.$$

The proof of the wave equation for $\vec{B}$ is similar.

4 True/False

Supply convincing reasoning for your answer. Assume all functions are infinitely differentiable unless stated otherwise.

(a) T F For every vector field $\vec{F}$ on $\mathbb{R}^3$, we have $\nabla \cdot (\nabla \times \vec{F}) = 0$.

Solution: TRUE. This follows by a direct computation in the same vein as those in Section 2 above.

(b) T F The divergence of a vector field is a scalar function, while the curl of a vector field is a vector field.

Solution: TRUE. This is immediate from the definitions.

(c) T F Every vector field $\vec{F}$ on $\mathbb{R}^3$ arises as the curl of some vector field $\vec{G}$.

Solution: FALSE. Because $\nabla \cdot (\nabla \times \vec{G}) = 0$ for all $\vec{G}$, we see that we must have $\nabla \cdot \vec{F} = 0$, which does not hold for all $\vec{F}$.

(d) T F Every function $f : \mathbb{R}^3 \to \mathbb{R}$ arises as the divergence of some vector field $\vec{F}$.

Solution: TRUE. For example, if we define

$$g(x, y, z) = \int_0^x f(t, y, z)\, dt,$$

then f is the divergence of the vector field $\vec{F} = \langle g, 0, 0 \rangle$ (by the fundamental theorem of calculus).

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.