Discussion 16 Worksheet Answers

Conservative vector fields and Green's theorem

Date: 10/29/2021

MATH 53 Multivariable Calculus

1 Conservative Vector Fields

For each of the following vector fields $\vec{F}$, either prove that $\vec{F}$ is conservative by finding a function f such that $\nabla f = \vec{F}$, or prove that f is not conservative. We can now do this systematically as showed in lecture.

1. $\vec{F}(x, y) = (xy + y^2)\,\vec{i} + (x^2 + 2xy)\,\vec{j}$

Solution: This is not conservative because

$$\frac{\partial}{\partial y}(xy + y^2) = x + 2y \neq 2x + 2y = \frac{\partial}{\partial x}(x^2 + 2xy).$$

2. $\vec{F}(x, y) = y^2 e^{xy}\,\vec{i} + (1 + xy)e^{xy}\,\vec{j}$

Solution: This is conservative.

$$\frac{\partial}{\partial y} y^2 e^{xy} = 2ye^{xy} + xy^2 e^{xy} = ye^{xy} + (y + xy^2)e^{xy} = \frac{\partial}{\partial x}(1 + xy)e^{xy}$$

and the domain is $\mathbb{R}^2$ which is open and simply connected. So $\exists f$ such that $\vec{F} = \nabla f$. So $f_x = y^2 e^{xy}$ and $f_y = (1 + xy)e^{xy}$. Integrating the first expression gives us $f(x, y) = ye^{xy} + g(y)$ and differentiating gives us $(1 + xy)e^{xy} = f_y = (1 + xy)e^{xy} + g'(y)$ so $g(y) = C \in \mathbb{R}$. Hence $f(x, y) = ye^{xy} + C$.

3. $\vec{F}(x, y, z) = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$

Solution: This is conservative. The potential is $f(x, y, z) = xyz + C$

2 Apply FTL

Find a function f such that $\vec{F} = \nabla f$ and use that to evaluate $\int_C \vec{F} \circ d\vec{r}$ along the curve C.

1. $\vec{F}(x, y) = x^2 y^3\,\vec{i} + x^3 y^2\,\vec{j}$ and $C : \vec{r}(t) = \langle t^3 - 2t, t^3 + 2t\rangle, 0 \leq t \leq 1$.

Solution: We see that $f(x, y) = x^3 y^3/3$ works as a potential function. Then the start and end point of C is $(0, 0)$ and $(-1, 3)$, respectively. Thus,

$$\int_C \vec{F} \circ d\vec{r} = \int_C \nabla f \circ d\vec{r} = f(-1, 3) - f(0, 0) = -9.$$

2. $\vec{F}(x,y) = \langle yz, xz, xy + 2z\rangle$ and C is the line segment from $(1,0,-2)$ to $(4,6,3)$.

Solution: We see that $f_x = yz$ so $f = xyz + g(y,z)$. Then $xz + g_y(y,z) = f_y = xz$ so $g_y(y,z) = 0$. This implies $g(y,z) = h(z)$. Then $xy + h'(z) = f_z = xy + 2z$ so $h(z) = z^2 + C$ for $C \in \mathbb{R}$. By FTL, the line integral evaluates to

$$f(4,6,3) - f(1,0,-2) = (4)(6)(3) + (3)^2 + C - (1)(0)(-2) - (-2)^2 - C = 81 - 4 = 77.$$

3. $\vec{F}(x,y,z) = \langle yze^{xz}, e^{xz}, xye^{xz}$ and $C : \vec{r}(t) = \langle t^2+1, t^2-1, t^2-2t\rangle, 0 \le t \le 2$.

Solution: It is easy to see that $f(x,y,z) = ye^{xz}$ is a potential function so the line integral evaluates to

$$f(5,3,0) - f(1,-1,0) = 3 - (-1) = 4.$$

3 More on line integrals

1. Show that the line integral is independent of path and evaluate it. $\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy$ and C is any path from $(1,0)$ to $(2,1)$.

Solution: The two functions have continuous partial derivatives on $\mathbb{R}^2$ and

$$\frac{\partial}{\partial y}(2xe^{-y}) = -2xe^{-y} = \frac{\partial}{\partial x}(2y - x^2e^{-y})$$

so the vector field with these functions as its components is conservative and hence the line integral is independent of path. It is easy to see that a potential function is $f(x,y) = x^2e^{-y} + y^2$ by running through the process as used in problems above. Then

$$\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy = f(2,1) - f(1,0) = 4/e + 1 - 1 = 4/e.$$

2. Find the work done by the force field $\vec{F} = \langle x^3, y^3\rangle$ in moving an object from $(1,0)$ to $(2,2)$.

Solution: We see that $\frac{\partial}{\partial y}(x^3) = 0 = \frac{\partial}{\partial x}(y^3)$ so there is a potential function for this vector field. The potential function turns out to be $f(x,y) = x^4/4 + y^4/4 + C$ where $C \in \mathbb{R}$. We can take $C = 0$. Thus

$$W = \int_C \vec{F} \circ d\vec{r} = f(2,2) - f(1,0) = 31/4.$$

4 Challenge

1. Consider the vector field

$$\vec{F}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle.$$

First show that $\partial P/\partial y = \partial Q/\partial x$. Then show that $\int_C \vec{F} \circ d\vec{r}$ is not independent of path. Does this contradict what we saw in lecture?

Solution: Firstly,

$$\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}.$$

Consider the two curves $C_1 : \vec{r}(t) = \langle \cos t, \sin t, 0 \le t \le \pi$ and $C_2 : \vec{r}(t) = \langle \cos t, \sin t, 2\pi \le t \le \pi$. Then

$$\int_{C_1} \vec{F} \circ d\vec{r} = \int_0^{\pi} \langle -\sin t, \cos t \rangle \circ \langle -\sin t, \cos t \rangle dt = \int_0^{\pi} dt = \pi$$

and $\int_{C_2} \vec{F} \circ d\vec{r} = \int_{2\pi}^{\pi} dt = -\pi$. Since these aren't equal, the line integral of F isn't independent of path. This doesn't contradict what we learned in class since the domain of $\vec{F}$ is $\mathbb{R}^2 \backslash \{(0,0\}$ isn't simply connected.

5 Line Integrals to Double Integrals

Use Green's theorem to convert each of the following line integrals $\int_C \vec{F} \circ d\vec{r}$ to double integrals. Then evaluate. All curves C are oriented counterclockwise.

1. C is the ellipse $x^2 + y^2/4 = 1$ and $\vec{F}(x,y) = \langle 2x - y, 3x + 2y \rangle$.

Solution: We have $\int_C \vec{F} \circ d\vec{r} = \iint_D Q_x - P_y = \iint_D 4dxdy = 4 \cdot \text{area(D)} = 8\pi$. Here and in all further solutions in this section, D is the region enclosed by C.

2. C is the circle $x^2 + y^2 = 1$ and $\vec{F}(x,y) = \frac{1}{3}\langle -y^3, x^3 \rangle$.

Solution: By Green's theorem, the line integral equals $\iint_{x^2+y^2 \le 1} x^2 + y^2 dxdy$. This integral is simplest in polar coordinates, where it becomes $\int_0^1 \int_0^{2\pi} r^3 d\theta dr = \pi/2$.

3. C is the triangle with vertices at $(0,0), (1,0), (0,1)$ and $\vec{F}(x,y) = \langle x^2 y, e^{y^2} + x \rangle$.

Solution: By Green's theorem, the line integral equals $\iint_D 1 - x^2 dxdy$. Firstly, $\iint_D dxdy = 1/2$. Next,

$$\iint_D x^2 dxdy = \int_0^1 \int_0^{1-y} x^2 dxdy = 1/12,$$

so altogether, the integral is $1/2 - 1/12 = 5/12$.

6 More on Green's Theorem

1. Let C be a simple, positively oriented, closed curve in $\mathbb{R}^2$. Using Green's theorem, check that $\int_C f(x)dx + g(y)dy = 0$ for arbitrary smooth functions f, g. Can you give an explanation without Green's theorem?

Solution: The line integral is zero by Green's theorem since $Q_x - P_y = 0$ in this situation. Without Green's theorem, let $\vec{r}(t) = \langle x(t), y(t)\rangle$ be a parameterization of C, with $0 \le t \le T$.
Then, $\int_C f(x)dx = \int_0^T f(x(t))x'(t)dt$. With the substitution $u = x(t)$, we see that this integral is zero, since $x(0) = x(T)$. The same is true for the other half of the integral. Alternatively, note that $\langle f(x), g(y)\rangle$ is conservative, since you may integrate each component separately to find a potential.

2. Consider the non-standard parameterization of the unit circle $x = \sin(t), y = \cos(t)$ with $0 \le t \le 2\pi$. Check that $\int_C xdy$ is not the area enclosed by C, as "promised" by Green's Theorem. What went wrong?

Solution: We have $\int_C xdy = \int_0^{2\pi} -\sin^2(t)dt = -\pi$ but the area enclosed by C is π. The problem is that this parameterization of the unit circle is oriented clockwise and Green's theorem requires a counterclockwise orientation. Reversing the direction of a path negates a line integral over that path.

3. Let C be the square centered at the origin with side length 4, oriented counterclockwise. Compute $\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$.

   Hint: Recall that the vector field $\langle P, Q\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$ satisfies $P_y = Q_x$ but is not conservative because $\int_\gamma Pdx + Qdy = 2\pi$ where γ is the unit circle, oriented counterclockwise.

Solution: If D is the region between C and γ, then $\int_D Q_x - P_y dxdy = 0$ since the integrand is zero, so $0 = \int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy - \int_\gamma \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$, so we find that the desired integral is 2π.

4. Let $\vec{F}$ be the vector field in the previous problem. Explain why, using Green's theorem, if C is a simple positively oriented curve contained in the upper half plane $y > 0$, then $\int_C \vec{F} \circ d\vec{r} = 0$.

Solution: Since C is a simple curve in the upper half plane, it bounds a region in the upper half plane, where the vector field is defined (since $(x, y) \neq (0, 0)$). Using the fact that $Q_x = P_y$ for this vector field, Green's theorem tells us that $\int_C \vec{F} \circ d\vec{r} = 0$. In fact, the hypothesis that C is simple is not needed.

7 True/False

(a) T F If the vector field $\vec{F} = \langle P, Q, R\rangle$ is conservative and the components have continuous first-order partial derivatives then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Solution: **True.** Since the components have continuous first-order partial derivatives and the vector field is conservative, by Clairaut's theorem we have $\frac{\partial P}{\partial y} = f_{xy} = f_{yx} = \frac{\partial Q}{\partial x}$ and similarly for the other identities.

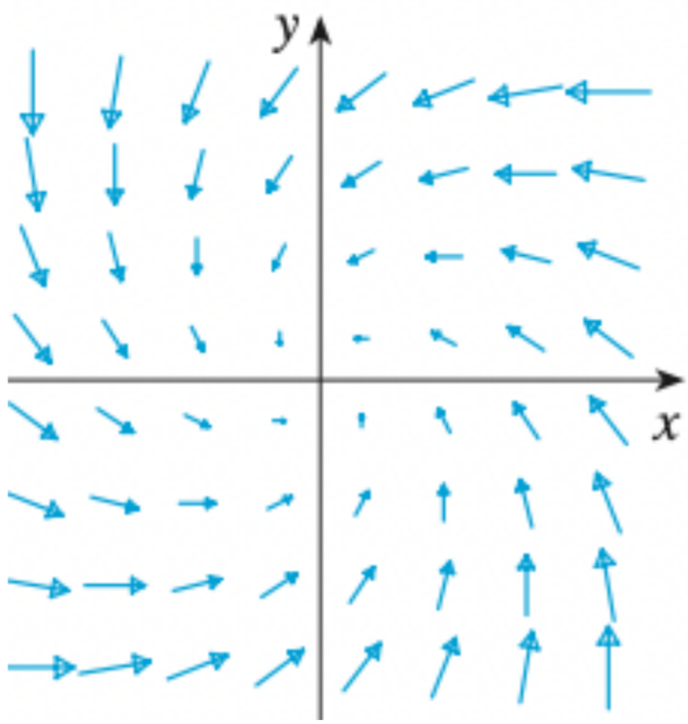
(b) T F You're asked to find the curve that requires the least work for the force field $\vec{F}$ to a move a particle from point to another point. You find that $\vec{F}$ is conservative. Then there a unique path satisfying this request.

Solution: False. The vector field being conservative means the work integral is path independent so any path from one of those points to the other will suffice.

(c) T F The line integral $\int_C y dx + x dy + xyz dz$ is path independent.

Solution: False. Using the first T/F question we see that $\partial P/\partial z = 0 \neq yz = \partial R/\partial x$ so the vector field $\langle y, x, xyz \rangle$ is not conservative so the integral is not path independent.

(d) T F The following vector field is conservative.



Solution: False. We know if the vector field is conservative then any line integral along a closed path will be zero. If we take C to be unit circle centered at the origin oriented ccw. All of the vector fields that start on C are in the direction of the motion along C so the line integral will be positive. Hence the vector field is not conservative.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.