Discussion 15 Worksheet Answers

Vector fields and line integrals

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MATH 53 Multivariable Calculus

1 Plotting Vector Fields

Create a plot of each of the following vector fields.

1. $\vec{F}(x, y) = x\,\vec{i} + y\,\vec{j}$
2. $\vec{F}(x, y) = x\,\vec{i} + x\,\vec{j}$
3. $\vec{F}(x, y) = -\dfrac{y}{x^2 + y^2}\,\vec{i} + \dfrac{x}{x^2 + y^2}\,\vec{j}$

2 Gradient Vector Fields

For each of the following functions, compute the gradient vector field, and draw this vector field in the plane. How does this vector field relate to the level sets of f?

1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = x^2 - y^2$
3. $f(x, y) = x - y^2$

3 Conservative Vector Fields

For each of the following vector fields $\vec{F}$, either prove that $\vec{F}$ is conservative by finding a function f such that $\nabla f = \vec{F}$, or prove that f is not conservative.

1. $\vec{F}(x, y) = x\,\vec{i} + y\,\vec{j}$

   Solution: This is conservative. To see this, write $\vec{F} = \nabla f$, and try to guess f. We get $f_x = x$ and $f_y = y$. Hence f_x has antiderivative $\frac{1}{2}x^2$ and f_y has antiderivative $\frac{1}{2}y^2$. These are not the same but we can add them and since each variable only appears in one of the atiderivatives, $f(x, y) = \frac{1}{2}(x^2 + y^2)$ satisfies for conditions.

2. $\vec{F}(x, y) = x\,\vec{i} + x\,\vec{j}$

   Solution: This is not conservative. If it were, we could write $\vec{F} = \nabla f$ for some $f(x, y)$. We would then have $f_x = x$ and $f_y = x$. But then we could compute

   $$f_{xy} = 0 \neq 1 = f_{yx},$$

   contradicting Clairaut's theorem.

3. $\vec{F}(x,y,z) = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$

Solution: This is conservative. To see that, we need to find an f such that $f_x = yz$, $f_y = xz$ and $f_z = xy$. But integrating each of these produces $xyz + C$ as a possible antiderivative, hence $f(x,y,z) = xyz$ is a potential for $\vec{F}$.

4. $\vec{F} = xz\,\vec{i} + yz\,\vec{j} + xy\,\vec{k}$

Solution: This is not conservative. If we write $\vec{F} = \nabla f$, then we compute $f_{xz} = x$, but $f_{zx} = y$, contradicting Clairaut's theorem.

4 Line integrals of functions

Compute the following line integrals:

1. $\int_C x\,ds$ where C is the graph of $f(x) = \frac{1}{2}x^2$ going from $x = 0$ to $x = 2$.

Solution: We can parametrize the graph using $x = t, y = f(t) = \frac{1}{2}t^2, 0 \leq t \leq 2$. Using this we get $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1+t^2}$. Now we compute, using the substitution $u = 1 + t^2$, $du = 2t\,dt$

$$\int_C x\,ds = \int_0^2 t\sqrt{1+t^2}\,dt = \frac{1}{2}\int_1^5 \sqrt{u}\,du = \frac{1}{2}\frac{2}{3}(5^{3/2} - 1^{3/2}) = \frac{5\sqrt{5}-1}{3}$$

2. $\int_C xy^4\,ds$ where C is the right half of the unit circle.

Solution: We know that C can be parametrized by $x = \cos t, y = \sin t, -\pi/2 \leq t \leq \pi/2$. Now using the definition of line integrals and $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\sin^2 t + \cos^2 t}$ we obtain

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} \cos t \sin^4 t\,dt = \int_{-1}^{1} u^4\,du = \frac{2}{5}$$

3. $\int_C x^2 y\,ds$ in 3D where C is given by $x = \cos t, y = \sin t, z = t, 0 \leq t \leq \pi/2$.

Solution: First compute $\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}$. Now we ca just compute

$$\int_C x^2 y\,ds = \int_0^{\pi/2} \cos^2 t \sin t \sqrt{2}\,dt = \sqrt{2}\int_0^1 u^2\,du = \frac{\sqrt{2}}{3}$$

5 Line integrals of vector fields

Compute the following line integrals:

1. $\int_C y^2\,dx + x^2\,dy$ where C is the line segment from $(1,0)$ to $(4,1)$.

Solution: We can parametrize C by $x = 1 + 3t, y = t, 0 \le t \le 1$. Hence

$$\int_C y^2\,dx + x^2\,dy = \int_0^1 t^2 \cdot 3\,dt + (1+3t)^2\,dt = \int_0^1 12t^2 + 6t + 1\,dt = 4 + 3 + 1 = 8$$

2. $\int_C x\,dx + y\,dy + z\,dz$ where C is the straight line connecting $(0,0,0)$ to $(1,2,3)$. Can you figure out what the integral will be when the endpoint of C is an arbitrary point (x_0, y_0, z_0) ?

Solution: We can parametrize this path by $x = t, y = 2t, z = 3t, 0 \le t \le 1$ obtaining

$$\int_C x\,dx + y\,dy + z\,dz = \int_0^1 t\,dt + 2t \cdot 2\,dt + 3t \cdot 3\,dt = \int_0^1 14t\,dt = 7$$

For the general case we can use the parametrization $x = x_0 t, y = y_0 t, z = z_0 t, 0 \le t \le 1$ and redoing the calculation above shows that

$$\int_C x\,dx + y\,dy + z\,dz = \int_0^1 ({x_0}^2 + {y_0}^2 + {z_0}^2)t\,dt = \frac{1}{2}\left({x_0}^2 + {y_0}^2 + {z_0}^2\right)$$

3. $\int_C \vec{F} \circ d\vec{r}$ where $\vec{F}(x,y) = (y,1)$ and C is the unit circle, traversed counterclockwise. Can you say something about the integral of $\vec{F_2}(x,y) = (y,0)$ along the same curve without doing another computation?

Solution: We parametrize the circle by $\vec{r}(t) = \langle \cos t, \sin t \rangle, 0 \le t \le 2\pi$ and obtain using $\vec{r}'(t) = \langle -\sin t, \cos t \rangle$:

$$\int_C \vec{F} \circ d\vec{r} = \int_0^{2\pi} -\sin^2 t + \cos t\,dt = -\frac{1}{2}2\pi = -\pi$$

We don't need to compute $\int_C \vec{F_2} \circ d\vec{r}$ again explicitly because $\vec{F_2}(x,y) - \vec{F}(x,y) = \langle 0, 1 \rangle = \nabla f$ for $f(x,y) = y$ is a conservative vector field. Hence $\int_C \vec{F_2} \circ d\vec{r} = \int_C \vec{F} \circ d\vec{r} + \int_C \nabla f \circ d\vec{r} = \pi + f(1,0) - f(1,0) = \pi$.

6 Challenge

1. Is the vector field

$$\vec{F}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

conservative (considered over the region R consisting of all points in $\mathbb{R}^2$ other than $(0,0)$)?

Solution: No it is not: Consider the line integral $\int_C \vec{F} \circ d\vec{r}$ where the curve C is the unit circle traversed counterclockwise, i.e. $x = \cos t, y = \sin t$. This is a path connecting the point $(1,0)$ to $(1,0)$ We compute

$$\int_C \vec{F} \circ d\vec{r} = \int_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{\sin^2 t\,dt + \cos^2 t\,dt}{\sin^2 t + \cos^2 t} = 2\pi$$

But if $\vec{F}$ were conservative i.e. $F = \nabla f$ for some (scalar) function f then we would have $\int_C \vec{F} \circ d\vec{r} = f(1,0) - f(1,0) = 0$ because C starts and ends at the point $(1,0)$.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.