Discussion 14 Worksheet Answers

Spherical coordinates and general changes of variables

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MATH 53 Multivariable Calculus

1 Spherical coordinates

1. Solve for ρ, ϕ, θ in terms of x, y, z. That is, find the inverse of the spherical coordinate mapping. Warning: you may need casework.

> **Solution:** First of all $\rho = \sqrt{x^2+y^2+z^2}$ by the Pythagorean theorem. Next, $\cos\phi = z/\rho = z/\sqrt{x^2+y^2+z^2}$ and we get $\phi = \arccos(z/\sqrt{x^2+y^2+z^2})$.
> Finally, $\tan(\theta) = y/x$ as long as $x \neq 0$. So, if (x, y) is in the first quadrant, then $\theta = \arctan(y/x)$. If (x, y) is in the second or third quadrant, then $\theta = \arctan(y/x) + \pi$ and if (x, y) is in the fourth quadrant, then $\theta = \arctan(y/x) + 2\pi$
> If $x = 0$, then θ will be $\pi/2$ or $3\pi/2$ according to whether $y > 0$ or $y < 0$. There is no well-defined angle θ when $x = y = 0$.

2. Describe the following surfaces (defined by Cartesian coordinates) in terms of spherical coordinates).

   $x = \sqrt{3}y$.

> **Solution:** This is equivalent to $\theta = \arctan(1/\sqrt{3}) = \pi/6$ OR $\theta = 7\pi/6$.

   $z^2 = x^2 + y^2$.

> **Solution:** By geometric reasoning, the answer is $\phi = \pi/4$ or $\phi = 3\pi/4$. This may also be done algebraically.

   $x^2 + y^2 + z^2/4 = 1$.

> **Solution:** By substitution, we get $\rho^2\sin^2\phi + \rho^2\cos^2\phi/4 = 1$.

3. Find the volume of the region bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 1$.

> **Solution:** The region enclosed is given by $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/3$ and $\sec\phi \le \rho \le 2$.
> So, the volume is (using $\sec'(\phi) = \sec\phi\tan\phi$)
>
> $$\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho d\phi d\theta = 2\pi \int_0^{\pi/3} \frac{1}{3}\sin(\phi)(8 - \sec^3(\phi))d\phi = 5\pi/3.$$

4. Compute the following integral over the region R lying above the cone $z^2 = x^2 + y^2$ and below the unit sphere

$$\iiint_R z^2 \, dV$$

Solution: The region is described by $0 \le \rho \le 1, 0 \le \phi \le \pi/4$ and $0 \le \pi \le 2\pi$, so

$$\iiint_R z^2\, dV = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} (\rho \cos\phi)^2 \sin\phi \rho^2\, d\theta\, d\phi\, d\rho \quad (1)$$

$$= 2\pi \int_0^1 \rho^4 \int_0^{\pi/4} \sin\phi \cos^2\phi\, d\phi\, d\rho \quad (2)$$

$$= \frac{2\pi}{5}\frac{1}{12}(4-\sqrt{2}) = \frac{\pi}{60}(4-\sqrt{2}) \quad (3)$$

5. Let d be a real number and consider the improper integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{(x^2+y^2+z^2)^d}.$$

For which values of d does this integral converge? Compute the integral for the values of d that make it converge.
Hint: As a first step, check that the region of integration is a sphere.

Solution: In spherical coordinates, this integral becomes

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 \frac{1}{\rho^{2d}} \rho^2 \sin\phi d\rho d\phi d\theta = 4\pi \int_0^1 \rho^{2-2d} d\rho.$$

This limit exists and equals $\frac{4\pi}{3-2d}$ when $d < 3/2$. Otherwise, the integral does not converge.

2 Calculating the Jacobian

Find the absolute value of the Jacobian determinant for each of the following changes of coordinates.

1. $x = au + bv$ and $y = cu + dv$.

Solution: We take the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which is $ad - bc$, so take absolute values to get $|ad - bc|$.

2. $x = u^2 - v^2$ and $y = 2uv$.

Solution: We compute $\det \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} = 4(u^2 + v^2)$, which is non-negative, so we do not need absolute values.

3. $x = e^u \cos(v)$ and $y = e^u \sin(v)$.

Solution: We compute $\det \begin{pmatrix} e^u \cos(v) & -e^u \sin(v) \\ e^u \sin(v) & e^u \cos(v) \end{pmatrix} = e^{2u}$. This is always positive.

4. $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$. Note that this transformation is its own inverse, in the sense that we can solve $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$. Also check that $(x^2 + y^2)(u^2 + v^2) = 1$.

Solution: We compute $\det \begin{pmatrix} \frac{1}{u^2+v^2} - \frac{2u^2}{(u^2+v^2)^2} & \frac{-2uv}{(u^2+v^2)^2} \\ \frac{2uv}{(u^2+v^2)^2} & \frac{-1}{u^2+v^2} + \frac{2v^2}{(u^2+v^2)^2} \end{pmatrix} = \frac{1}{(u^2+v^2)^2}$, which, as before, is always positive.

3 Integrating with change of variables

1. Consider the region $\mathcal{R}$ in the plane: $3x^2 + 4xy + 3y^2 \leq 1$.

   Describe the transformed region using the change of variables $x = v - u$ and $y = u + v$.

   Find the area of $\mathcal{R}$.

   Solution: We have $3(v-u)^2 + 4(v^2 - u^2) + 3(u+v)^2 = 2u^2 + 10v^2 \leq 1$. This is an ellipse with axes of length $1/\sqrt{2}$ and $1/\sqrt{10}$, so has area $\pi/\sqrt{20}$. The Jacobian of this transformation is 2.
   Now, $A = \int_{\mathcal{R}} dxdy = \int_{\mathcal{R}'} 2dudv = 2\pi/\sqrt{20} = \pi/\sqrt{5}$, where $\mathcal{R}'$ is the ellipse in the uv-plane.

2. Let D be the annulus $1 \leq x^2 + y^2 \leq 4$ and consider the integral

$$\iint_D \frac{1}{(x^2+y^2)^2} e^{\frac{x}{x^2+y^2}} dxdy.$$

   Perform the change of variables $x = \frac{u}{u^2+v^2}, y = \frac{-v}{u^2+v^2}$ to simplify the integral, but do not evaluate.

   Solution: From our work on this substitution in the first problem, we know that the new region of integration is $D' : 1/4 \leq u^2 + v^2 \leq 1$ and the integral becomes

$$\iint_{D'} (u^2+v^2)^2 e^u \frac{1}{(u^2+v^2)^2} dudv = \iint_{D'} e^u dudv.$$

4 True/False

Supply convincing reasoning for your answer.

(a) T F If the Jacobian of a transformation $x = x(u,v), y = y(u,v)$ is always non-zero, then the transformation is one-to-one.

Solution: False: Consider the transformation $x = u^2 - v^2, y = 2uv$, defined on the whole uv-plane except for the point $u = v = 0$. The Jacobian is positive, but $(1,1)$ and $(-1,-1)$ map to the same point.

(b) T F The image of a rectangle in the plane under the transformation $x = 2u, y = -2v$ will be another rectangle.

Solution: True: Scaling a rectangle by a factor of 2 and reflecting will result in another rectangle.

(c) T F There is a point with spherical coordinates $\rho = 1/2, \phi = 3\pi/2, \theta = \pi/2$.

Solution: False: The angle ϕ can be at most π (otherwise points will be double-counted).

(d) T F The "ρ" in spherical coordinates equals the "r" in cylindrical coordinates.

Solution: False: ρ represents the distance from the origin, while r represents the distance from the z-axis.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.