Discussion 13 Worksheet Answers

Double integrals in polar coordinates and surface areas of graphs

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MATH 53 Multivariable Calculus

1 Double integral practice

Compute these integrals:

(a) $\iint_D x\cos y dA$ where D is bounded by $y = 0, y = x^2, x = 1$;

> **Solution:** $\int_0^1 \int_0^{x^2} x\cos y dydx = \int_0^1 x\sin y|_0^{x^2} dx = \int_0^1 x\sin x^2 dx = -\frac{1}{2}\cos(x^2)|_0^1 = \frac{1}{2}(1-\cos 1)$.

(b) $\iint_D 2x - ydA$ where D is bounded by the circle with center at the origin and radius 2.

> **Solution:** $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2x - ydydx = \int_{-2}^2 2xy - \frac{y^2}{2}|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \int_{-2}^2 4x\sqrt{4-x^2} = 0$ since $4x\sqrt{4-x^2}$ is an odd function.

(c) Find the volume of the solid under the surface $z = xy$ and above the triangle with vertices $(1, 1), (4, 1), (1, 2)$.

> **Solution:** After finding the equations bounding this triangle, we set up the integral and solve. The bounding lines are $x = 1, y = 1, x + 3y = 7$ so the volume is
>
> $$\int_1^2 \int_1^{7-3y} xydxdy = \int_1^2 \frac{1}{2}x^2y|_1^{7-3y}dy = \frac{1}{2}\int_1^2 48y - 42y^2 + 9y^3 dy = \frac{31}{8}$$

(d) Find the volume exnclosed by the cylinders $z = x^2, y = x^2$ and the planes $z = 0$ and $y = 4$.

> **Solution:** The set up is $\int_{-2}^2 \int_{x^2}^4 x^2 dydx = \int_{-2}^2 4x^2 - x^4 dx = \frac{128}{15}$.

(e) Find the volume of the solid by subtracting two volumes. The solid is enclosed by the parabolic cylinders $y = 1 - x^2, y = x^2 - 1$ and the planes $x + y + z = 2, 2x + 2y - z + 10 = 0$.

> **Solution:** The region of integration is bounded by the curves $y = 1 - x^2, y = x^2 - 1$ which intersect at $x = \pm 1$ with $1 - x^2 \geq x^2 - 1$ on $[-1, 1]$. Within this region, $z = 2x + 2y + 10$ is above $z = 2 - x - y$ so we have
>
> $$V = \int_{-1}^1 \int_{x^2-1}^{1-x^2} 2x + 2xy + 10dydx - \int_{-1}^1 \int_{x^2-1}^{1-x^2} 2 - x - ydydx = \int_{-1}^1 \int_{x^2-1}^{1-x^2} 3x + 3y + 8dydx$$
>
> $$= \int_{-1}^1 3xy + \frac{3}{2}y^2 + 8y|_{x^2-1}^{1-x^2} dx = \int_{-1}^1 -6x^3 - 16x^2 + 6x + 16dx = \frac{64}{3}.$$

2 Polar Integration

Remember dA becomes $r dr d\theta$.

(a) $\iint_D x^2 y dA$ where D is the top half of the disk with center the origin and radius 5;

Solution: The region is $D = \{(r, \theta) \mid 0 \leq r \leq 5, 0 \leq \theta \leq \pi\}$. Then

$$\iint_D x^2 y dA = \int_0^\pi \int_0^5 (r\cos\theta)^2 r \sin\theta r dr d\theta = \left(\int_0^\pi \cos^2\theta \sin\theta d\theta\right)\left(\int_0^5 r^4 dr\right) = \frac{1250}{3}.$$

(b) $\iint_D e^{-x^2-y^2} dA$ where D is the region bounded by the semicircle $x = \sqrt{4-y^2}$ and the y-axis.

Solution:

$$\iint_D e^{-x^2-y^2} dA = \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} r dr d\theta = \pi\left(-\frac{1}{2}e^{-r^2}\Big|_0^2\right) = \frac{\pi}{2}(1-e^{-4}).$$

3 Surface Areas

Parametrize the following surfaces in an appropriate way (if they are not already parametrized) and compute their normal vectors and area.

(a) The portion of the elliptic paraboloid $z = x^2 + y^2$ lying over the unit disk.

Solution: This surface is the graph of $f(x, y) = x^2 + y^2$, so we know that

$$\vec{N} = \langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle = \langle -f_x, -f_y, 1\rangle = \langle -2x, -2y, 1\rangle.$$

The area is computed by the following integral over the unit disk D, which we compute in polar coordinates and using the substitution $u = 1 + 4r^2$

$$\int_D |\vec{N}|\, dA = \int_D \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{1+4r^2}\, r\, dr\, d\phi$$
$$= 2\pi \int_1^5 \sqrt{u}\frac{1}{8}\, du = \frac{\pi}{6}\left(5^{3/2} - 1\right)$$

(b) The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

Solution: This is the graph of $f(x, y) = xy$, so

$$\vec{N} = \langle -f_x, -f_y, 1\rangle = \langle y, x, 1\rangle.$$

Restricting the surface to the part inside the cylinder corresponds to restricting the domain of f to the unit disk D. The area of the surface is given by

$$A = \iint_D \sqrt{1 + x^2 + y^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2+1} r\, dr\, d\theta = \frac{2\pi}{3}\left(2\sqrt{2} - 1\right)$$

(The computation of the integral is analogous to problem 3.(a)).

4 Triple Integration

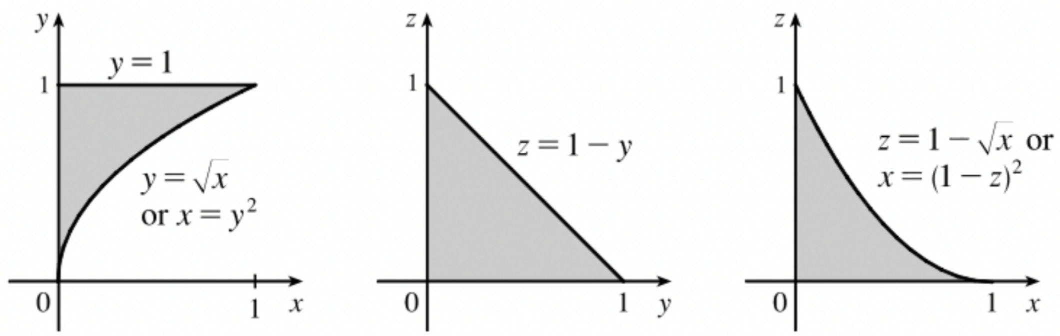
Change the order of integration for these integrals. Sketching the region of integration might be helpful.

(a) Rewrite the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{y-1} f(x,y,z)dzdydx$$

as the equivalent iterated integral in the five other orders.

Solution: The diagrams show the projections of the solid onto the coordinate planes.



Hence

$$\begin{aligned}\int_0^1 \int_{\sqrt{x}}^1 \int_0^{y-1} f(x,y,z)dzdydx &= \int_0^1 \int_0^{y^2} \int_0^{y-1} f(x,y,z)dzdxdy \\ &= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)dxdydz \\ &= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)dxdzdy \\ &= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)dydzdx \\ &= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)dydxdz\end{aligned}$$

5 Challenge

(a) Find the volume of a right pyramid by setting up a triple integral. (Hint: Place 3 vertices on the coordinate axes and the fourth at the origin and use the plane equation.)

Solution: Following the hint we can set up a generic plane containing the points $(a, 0, 0), (0, b, 0), (0, 0, c)$. The equation for this plane is given by $z = c(1 - x/a - y/b)$. Let D be the projection to the xy-plane defined by $bx + ay = ab$. Then the volume is

$$
\begin{aligned}
V = \iint_D \int_0^{c(1-x/a-y/b)} dzdA = \iint_D c(1 - x/a - y/b)dA \\
&= cA(D) - \frac{c}{a}\int_0^a \int_0^{b-bx/a} xdydx - \frac{c}{b}\int_0^b \int_0^{a-ay/b} ydxdy \\
&= cA(D) - \frac{c}{a}\int_0^a bx - bx^2/adydx - \frac{c}{b}\int_0^b ay - ay^2/bdxdy \\
&= \frac{abc}{2} - \frac{abc}{6} - \frac{abc}{6} \\
&= \frac{abc}{6} \\
&= cA(D)/3
\end{aligned}
$$

which is (1/3) times the area of base times the height.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.