Discussion 12 Worksheet Answers

Double integrals

Date: 10/11/2021

MATH 53 Multivariable Calculus

1 Double Integrals

Use geometric arguments to find the values of the following integrals.

1. $\iint_{[0,a]\times[0,b]} c dA$ where a, b, c are all real positive constants.

   Solution: This is the volume of a rectangular prism with side lengths a, b, and c, so the answer is abc.

2. $\iint_{x^2+y^2\leq 1} \sqrt{1-x^2-y^2} dA$

   Solution: This is the volume of the upper half-ball of the ball $x^2+y^2+z^2 \leq 1$. The volume of a ball is $(4/3)\pi R^3$, so our volume is $(2/3)\pi$.

3. $\iint_{x^2+y^2\leq 1}(1-\sqrt{x^2+y^2}) dA$

   Solution: This is the volume of a cone of height 1 and radius 1. The volume of a cone is $\pi rh/3$, so our integral gives $\pi/3$.

4. $\iint_{|x|+|y|\leq 1}(1-|x|-|y|) dA$

   Solution: This is a square pyramid with base side length $\sqrt{2}$ and height 1. The volume of a square pyramid is $bh/3$, so our integral gives $\sqrt{2}^2/3 = 2/3$.

2 Changing the order of integration

Change the order of integration for these integrals. Sketching the region of integration might be helpful.

(a) $\displaystyle\int_0^1 \int_0^y f(x,y)\, dx\, dy$

Solution: The region can be described by $0 \leq y \leq 1, 0 \leq x \leq 1$ and $x \leq y$, so it is equivalent to $0 \leq x \leq 1, x \leq y \leq 1$. Therefore the integral above is equal to

$$\int_0^1 \int_x^1 f(x,y)\, dy\, dx.$$

(b) $\displaystyle\int_0^{\pi/2} \int_0^{\cos x} f(x,y)\, dy\, dx$

Solution: $0 \le \cos x \le 1$ for $0 \le x \le \pi/2$ so this region can be described by $0 \le x \le \pi/2, 0 \le y \le 1$ and $y \le \cos x$. We also know that arccos is an decreasing function, i.e. for $a < b$ we have $\arccos a > \arccos b$. Therefore this region is equivalent to $0 \le y \le 1, 0 \le x \arccos y$ and switching the order of integration gives

$$\int_0^1 \int_0^{\arccos y} f(x,y)\, dx\, dy.$$

(c) $\displaystyle\int_1^2 \int_0^{\ln x} f(x,y)\, dy\, dx$

Solution: The domain of integration is $1 \le x \le 2, 0 \le y \le \ln 2,\ y \le \ln x$. The last inequality is equivalent to $e^y \le x$ and so switching the order of integration gives

$$\int_0^{\ln 2} \int_{e^y}^2 f(x,y)\, dx\, dy.$$

3 Double integral practice

Compute these integrals:

(a) $\displaystyle\int_0^1 \int_0^v \sqrt{1-v^2}\, du\, dv$

Solution: Evaluating the inner integral gives

$$\int_0^1 v\sqrt{1-v^2}\, dv = \frac{1}{2}\int_0^1 \sqrt{s}\, ds = \frac{1}{3}.$$

In the second step we used the substitution $s = 1 - v^2$, so $ds = -2v\, dv$.

(b) $\displaystyle\iint_D dA$ where $D = \{(x,y) \mid x^2 + y^2 \le 1\}$ (You can know the answer before doing the computation.)

Solution: This integral computes the area of D which is the unit circle, so we already know that the answer is going to be π. For the actual computation we write $D = \{(x,y) \mid -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$ as a region of type I, so the integral becomes

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dy\, dx = \int_{-1}^1 2\sqrt{1-x^2}\, dx = 2\int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2\phi}\cos\phi\, d\phi$$
$$= 2\int_{-\pi/2}^{\pi/2} \cos^2\phi\, d\phi = \pi.$$

(c) $\displaystyle\iint_D x\, dA$ where $D = \{(x,y) \mid 0 \le x \le \pi, 0 \le y \le \sin x\}$

Solution: D is a type I region so we compute the integral as

$$\int_0^\pi \int_0^{\sin x} x\,dy\,dx = \int_0^\pi x\sin x\,dx = -\,x\cos x\,|_0^\pi + \int_0^\pi \cos x\,dx = \pi.$$

(d) $\displaystyle\iint_D (x+y)\,dA$ where D is bounded by $y=\sqrt{x}$ and $y=x^2$

Solution: D is a type I region since the y-values lie between the graphs of two function of x. To know the bounds for x we first need to find where the two graphs intersect, i.e. solve $\sqrt{x}=x^2$. Squaring both sides turns this into $x^2=x^4$ or equivalently $x^2(1-x^2)=0$ which has solutions $-1,0,1$. Since $\sqrt{x}$ is only defined for $x\geq 0$ we see that the graphs intersect at $x=0,1$, so these are our bounds for x. Now we compute the double integral to be

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 x(\sqrt{x}-x^2)+\frac{1}{2}(x-x^4)\,dx = \frac{2}{5}-\frac{1}{4}+\frac{1}{4}-\frac{1}{10}=\frac{3}{10}.$$

(e) $\displaystyle\int_0^1 \int_{4y}^4 e^{x^2}\,dx\,dy$

Solution: The function e^{x^2} has no antiderivative that we can express with familiar functions so we try changing the order of integration. The domain of integration is given by $0\leq y\leq 1, 4y\leq x\leq 4$, which can also be expressed as $0\leq y\leq 1, 0\leq x\leq 4$ and $4y\leq x$. From this second form we see that the region is equivalent to $0\leq x\leq 4, 0\leq y\leq x/4$. Hence the integral can be computed as

$$\int_0^4 \int_0^{x/4} e^{x^2}\,dy\,dx = \int_0^4 \frac{x}{4}e^{x^2}\,dx = \int_0^{16} \frac{1}{8}e^s\,ds = \frac{1}{8}\left(e^{16}-1\right).$$

We used the substitutions $x^2=s$.

(f) $\displaystyle\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,dx\,dy$

Solution: The inner integral looks hard so we try switching the order of integration. The region of integration is the rectangle $[0,\pi/2]\times[0,1]$ with the extra constraint $\arcsin y\leq x$, or equivalently $y\leq \sin x$. Hence our integral is equal to

$$\begin{aligned}\int_0^{\pi/2} \int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\,dy\,dx &= \int_0^{\pi/2} \sin x\cos x\sqrt{1+\cos^2 x}\,dx\\ &= \frac{1}{2}\int_0^1 \sqrt{1+s}\,ds\\ &= \frac{1}{3}\left(2\sqrt{2}-1\right).\end{aligned}$$

4 Challenge

Compute

$$I = \iint_D \sqrt{1 - x^2 - y^2}\, dA$$

where D is the unit circle without using polar coordinates or geometric arguments. What is the solid whose volume we are computing here?

Solution: I is the volume of a hemisphere. To compute this we are going to need the integral

$$\int_{-a}^{a} \sqrt{a^2 - s^2}\, ds = \int_{-\pi/2}^{\pi/2} \sqrt{a^2 - a^2 \sin^2 \phi}\, a \cos \phi \, d\phi = a^2 \int_{-\pi/2}^{\pi/2} \cos^2 \phi \, d\phi = a^2 \frac{\pi}{2}$$

Here we did a substitution $s = a \sin \phi$. Now we can proceed to compute I:

$$\begin{aligned} I &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{\left(\sqrt{1-x^2}\right)^2 - y^2}\, dy\, dx \\ &= \int_{-1}^{1} \frac{\pi}{2} \left(1 - x^2\right) dx \\ &= \frac{\pi}{2}(2 - 2/3) = \frac{2\pi}{3}. \end{aligned}$$

5 True/False

Supply convincing reasoning for your answer.

(a) T F If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then f is the derivative of $\iint f dA$.

Solution: FALSE. Although this is true for single-variable functions, it does not even make sense for functions of several variables, because we don't have a definition for a single "derivative" of something like $\iint f dA$.

(b) T F In some simple cases, computing double integrals reduces to computing the volumes of well-known solids.

Solution: TRUE. For example, the double integrals in problem **??** above can be computed using this method.

(c) T F $\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy\, dx = \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx\, dy$ by Fubini's theorem.

(Hint: $\frac{d}{dy} \frac{y}{x^2+y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$*)*

(Hint 2: I wouldn't be giving the above hint if you didn't have to compute the integrals...)

Solution: **FALSE.** We compute the LHS:

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dy\, dx = \int_0^1 \left. \frac{y}{x^2+y^2} \right|_{y=0}^{y=1} dx$$
$$= \int_0^1 \frac{1}{1+x^2}\, dx$$
$$= \arctan 1 - \arctan 0 = \pi/4$$

Now we compute the RHS:

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dx\, dy = \int_0^1 \left. \frac{-x}{x^2+y^2} \right|_{x=0}^{x=1} dy$$
$$= \int_0^1 \frac{-1}{1+y^2}\, dy$$
$$= -\arctan 1 + \arctan 0 = -\pi/4$$

What is this sorcery? Did Guido Fubini lie to us? No! We can't apply Fubini's theorem here because the integrand becomes infinite around zero and therefore isn't continuous on $[0,1] \times [0,1]$.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.