Discussion 11 Worksheet Answers
Lagrange Multipliers
Date: 10/4/2021

MATH 53 Multivariable Calculus

1 Lagrange Multipliers

1. Find the extreme values of the function $f(x, y) = 2x + y + 2z$ subject to the constraint that $x^2 + y^2 + z^2 = 1$.

Solution: We solve the Lagrange multiplier equation: $\langle 2, 1, 2\rangle = \lambda\langle 2x, 2y, 2z\rangle$. Note that λ cannot be zero in this equation, so the equalities $2 = 2\lambda x, 1 = 2\lambda y, 2 = 2\lambda z$ are equivalent to $x = z = 2y$. Substituting this into the constraint yields $4y^2+y^2+4y^2 = 1$, so $y = \pm 1/3$. The max and min values occur at $(2/3, 1/3, 2/3)$ and $(-2/3, -1/3, -2/3)$, respectively, with function values ± 3.

2. Find the extreme values of the function $f(x, y) = y^2e^x$ on the domain $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

Solution: The gradient of this function is $(y^2e^x, 2ye^x)$, which is zero along the x-axis $y = 0$. Here the function value of 0 is a minimum, since $f(x, y) \geq 0$ everywhere.
On the boundary we have the Lagrange multiplier equation: $y^2e^x = 2\lambda x$ and $2ye^x = 2\lambda y$. We may assume $y \neq 0$ as we have already considered this case, and then we get $2y = x/y$, so $y^2 = 2x$. Together with the equation $x^2 + y^2 = 1$, we obtain $2 - x^2 = 2x$, so $x = \pm\sqrt{2} - 1$. We only need the "+" solution because the "-" one lies outside of the unit disk. We know $y^2 = 2x = 2(\sqrt{2}-1)$ and therefore the maximum value of f on the unit disk is

$$f(\sqrt{2} - 1, \pm\sqrt{2(\sqrt{2} - 1)}) = 2(\sqrt{2} - 1)e^{\sqrt{2}-1}$$

3. Use Lagrange multipliers to find the closest point(s) on the parabola $y = x^2$ to the point $(0, 1)$. How could one solve this problem without using any multivariate calculus?

Solution: We maximize the function $f(x, y) = x^2 + (y - 1)^2$ subject to the constraint $g(x, y) = y - x^2 = 0$.
We obtain the system of equations

$$\begin{aligned} 2x &= -2\lambda x \\ 2(y-1) &= \lambda \end{aligned}$$

Substituting the second equation into the first, we find $2x = -2(2(y - 1))x$, so either $x = 0$ or $y = 1/2$. In the first case, the point $(0, 0)$ is distance 1 from $(0, 1)$. In the second case, $(\pm\frac{1}{\sqrt{2}}, 1/2)$ is distance $\sqrt{1/2 + 1/4} = \sqrt{3/4} < 1$ from the point $(0, 1)$. These two points are the closest.
This problem could also be solved by minimizing the function $\sqrt{t^2 + (t^2 - 1)^2}$.

4. You have 24 square inches of cardboard and want to build a box (in the shape of a rectangular prism). Show that a 2" $\times$ 2" $\times$ 2" cube encloses the largest volume.

Solution: If x, y, z are the side lengths of the solid, then we have a constraint $xy + yz + zx = 12$ and want to optimize the function $f(x, y, z) = xyz$.
A maximum value must exist since the volume goes to zero if any of the side lengths do.
We have $yz = \lambda(y + z)$ and $xz = \lambda(x + z)$ and $xy = \lambda(x + y)$. Multiplying the first equation by x and the second by y and equating, we get $x\lambda(y + z) = xyz = y\lambda(x + z)$. All quantities are positive, so we may simplify to get $x(y+z) = y(x+z)$, which simplifies to $x = y$. Arguing similarly with the third equation, we find that all side lengths are equal.

5. Find the largest possible volume of a rectangular prism with edges parallel to the coordinate axes and all vertices lying on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(where $a, b, c > 0$.)

Solution: Let x, y, and z each be half of the side length pointing along the coordinate axes. Then the volume of the prism is $f(x, y, z) = 8xyz$. We want to maximize this subject to the constraint $g = 1$, where $g(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2$. Our Lagrange multiplier equation $\nabla f = \lambda \nabla g$ becomes

$$8yz = \frac{2\lambda x}{a^2}, \qquad 8xz = \frac{2\lambda y}{b^2}, \qquad 8xy = \frac{2\lambda z}{c^2}$$

If $\lambda = 0$ then at least one of x, y, and z must be zero, giving a total volume of zero. As this is clearly not maximal, we can ignore this case and assume $\lambda \neq 0$. Multiplying the first equation by $x/2\lambda$ gives $x^2/a^2 = xyz/2\lambda$. Let $k = xyz/2\lambda$; then we are just saying $x^2/a^2 = k$. Similarly, we obtain $y^2/b^2 = z^2/c^2 = k$. Plugging these into the equation for the ellipse gives $3k = 1$, so $k = 1/3$. Thus $x = \pm\frac{1}{\sqrt{3}}a$, and since x is a length, we should get $x = \frac{1}{\sqrt{3}}a$. Similarly, we obtain $y = \frac{1}{\sqrt{3}}b$ and $z = \frac{1}{\sqrt{3}}c$.

6. Use Lagrange multipliers to find the closest points to the origin on the hyperbola $xy = 1$.

Solution: We want to minimize $f(x, y) = x^2 + y^2$ subject to $g(x, y) = 1$, where $g(x, y) = xy$. Setting $\nabla f = \lambda \nabla y$, we obtain $2x = \lambda y$ and $2y = \lambda x$. If $\lambda = 0$, then $x = y = 0$, but $(0, 0)$ is not a point on the hyperbola, so we can ignore this case. So $\lambda \neq 0$, and we can write $y = 2x/\lambda$. Plugging this into $xy = 1$, we get $2x^2/\lambda = 1$, or $\lambda = 2x^2$. Taking this equation and plugging it into $2y = \lambda x$, we see $2y = 2x^3$, or $y = x^3$. Then $1 = xy = x^4$, so $x = \pm 1$. For $x = 1$ we solve $xy = 1$ to get $y = 1$; likewise, for $x = -1$ we get $y = -1$. It is geometrically obvious that these correspond to minima, so the closest points to the origin on $xy = 1$ are $(1, 1)$ and $(1, -1)$.

2 Lagrange multipliers with two constraints

1. Maximize and minimize $3x - y - 3z$ subject to $x + y - z = 1$ and $x^2 + 2z^2 = 1$.

Solution: Let $f = 3x - y - 3z$, $g = x + y - z$, $h = x^2 + 2z^2$. Then $\nabla f = (3, -1, -3)$, $\nabla g = (1, 1, -1)$, and $\nabla h = (2x, 0, 4z)$. Our Lagrange multiplier equation $\nabla f = \lambda \nabla g + \mu \nabla h$ splits into

$$3 = \lambda + 2\mu x, \qquad -1 = \lambda + 0, \qquad -3 = -\lambda + 4\mu z.$$

Hence $\lambda = -1$, and we can plug this in to the other equations to see $\mu = 2/x = -1/z$, so $x = -2z$. Plugging this into $x^2 + 2z^2 = 1$ gives $6z^2 = 1$ so $z = \pm 1/\sqrt{6}$, $x = \mp 2/\sqrt{6}$ (so x has the opposite sign of z). Plugging this into $x + y - z = 1$ shows $y = 1 + 3z$ and so (x, y, z) is either

$$(-2/\sqrt{6}, 1 + 3/\sqrt{6}, 1/\sqrt{6})$$

or

$$(2/\sqrt{6}, 1 - 3/\sqrt{6}, -1/\sqrt{6}).$$

Computing $3x - y - 3z$ for each shows that the former gives a minimum $(-1 - 2\sqrt{6})$ and the latter gives a minimum $(1 + 2\sqrt{6})$.

2. Maximize and minimize z subject to $x^2 + y^2 = z^2$ and $x + y + z = 24$.

Solution: This has no maximum or minimum. How do we see this? We show that when z is large (how large exactly we're about to see) then the system

$$\begin{aligned} x^2 + y^2 &= z^2 \\ x + y + z &= 24 \end{aligned}$$

has a solution (x, y). To check this we solve for y in the second equation and plug back into the first, obtaining

$$x^2 + (24 - x - z)^2 = z^2$$

which simplifies to

$$x^2 + (z - 24)x + (288 - 24z) = 0$$

This is a quadratic equation and we know that they have solutions when the discriminant[1] is greater or equal to zero. Here the discriminant is

$$(z - 24)^2 + 4(288 - 24z) = z^2 + 48z - 576$$

This describes a parabola that's "open from above" so when z is very large or very negative the discriminant will be positive, meaning that there are x, y such that (x, y, z) satisfies our constraints. So $f(x, y, z)$ can be arbitrarily large and arbitrarily small given our constraints.

[1] The discriminant of a quadratic equation $ax^2 + bx + c$ is $b^2 - 4ac$. That's the expression inside the square root in the quadratic formula.

3 Challenge

1. Using the method of Lagrange multipliers, prove the following inequality: if $x_1, \ldots, x_n$ are positive real numbers, then

$$\frac{n}{1/x_1 + \ldots + 1/x_n} \leq \sqrt[n]{x_1 \ldots x_n}$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$. The lefthand side is called the *harmonic mean* of the numbers $x_1, \ldots, x_n$ and the righthand side is called their *geometric mean.*

Solution:
We maximize the function $f(x_1, \ldots, x_n) = \frac{n}{1/x_1+\ldots+1/x_n} \leq \sqrt[n]{x_1 \ldots x_n}$ subject to the constraint that $g(x_1, \ldots, x_n) := x_1 \ldots x_n = C$ for a constant C.
Note that maximizing f is equivalent to minimizing the function $F(x_1, \ldots, x_n) = \frac{1}{x_1} + \ldots + \frac{1}{x_n}$. This function must obtain a minimum on the hypersurface $x_1 \ldots x_n = C > 0$ because this quantity tends to infinity as $\min(x_1, \ldots, x_n) \to 0$, so the minimum must occur at a point found by Lagrange multipliers (since the gradient of the constraint function is nonzero on its level set.)
For each k, we have

$$\frac{-1}{x_k^2} = \lambda x_1 \ldots \hat{x_k} \ldots x_n.$$

Where the hat over x_k indicates that it is omitted from the product. Rearranging,

$$-1 = \lambda x_1 \ldots x_k^2 \ldots x_n = C\lambda x_k.$$

Now, λ must be nonzero for this to hold, in which case we find that $x_1 = \ldots = x_n$ $(= \sqrt[n]{C})$, which we may check gives equality for the claimed inequality. By the previous reasoning, this must correspond to a minimum for F, or a maximum for f, so at any other point, the LHS is strictly smaller than the RHS.

2. As in problem 1.4., find the dimensions of the box enclosing the largest volume if the box has no top. Hint: try making a substitution before using Lagrange multipliers.

Solution: We want to maximize $f(x, y, z) = xyz$ subject to the constraint $xy + 2yz + 2xz = 24$. We make the substitution $u = xy, v = yz, w = xz$ so that we are maximizing uvw (which is $(xyz)^2$) subject to the constraint $u + 2v + 2w = 24$.
Now the Lagrange multiplier equations are $vw = \lambda$ and $uw = 2\lambda$ and $uv = 2\lambda$. The last two equations give $v = w$. The first two equations give $u = 2v$. In terms of x, y, z, this means $yz = xz$, so $y = x$, and similarly $x = 2z$. So the sides are in ratio $2 : 2 : 1$. Together with the original constraint $xy + 2yz + 2xz = 24$, we get $x = y = 2\sqrt{2}$ and $z = \sqrt{2}$

3. If $x_1, \ldots, x_n$ are real numbers, prove that

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leq n \left(\sum_{i=1}^{n} x_i^2\right).$$

Solution: Let $r = \sqrt{\sum_i x_i^2}$. Define functions $f(y_1, \ldots, y_n) = \sum_i y_i$ and $g(y_1, \ldots, y_n) = \sum_{i=1}^n y_i^2$. To show our desired inequality, it suffices to show that the maximum value of f on the sphere $g(y_1, \ldots, y_n) = r$ is at most $\sqrt{nr}$ (because then $f(x_1, \ldots, x_n) \leq \sqrt{nr}$, so $f(x_1, \ldots, x_n)^2 \leq nr$, which is exactly the inequality we are trying to show). So we optimize f subject to the constraint $g = r$.
To do this, we use Lagrange multipliers, and so we set $\nabla f(y_1, \ldots, y_n) = \lambda \nabla g(y_1, \ldots, y_n)$ for some scalar λ. Computing our gradients and plugging them in, we get $1 = 2\lambda y_i$ for each i. Thus we must have $y_i = 1/(2\lambda)$ for all i (since $\lambda = 0$ would lead to the equation $1 = 0$, which can't hold). Plugging these into the equation $g(y_1, \ldots, y_n) = r$, we obtain

$$r = \sum_{i=1}^{n} \frac{1}{4\lambda^2} = \frac{n}{4\lambda^2},$$

so $\lambda = \pm\frac{1}{2}\sqrt{n/r}$. It follows that $y_i = \pm\sqrt{r/n}$ for all i, so

$$\sum_i y_i = n \cdot \pm\sqrt{\frac{r}{n}} = \pm\sqrt{nr}.$$

The (global) maximum is clearly obtained when the sign here is $+$, so we see that the maximum value of f on the sphere $g = r$ is $\sqrt{nr}$, as needed.

4 True/False

Supply convincing reasoning for your answer.

(a) T F Any continuous function on the domain $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ will attain a maximum.

Solution: False: $f(x, y) = x$ is a counterexample.

(b) T F If $xye^x = \lambda y$ and $xye^x = \lambda x$, then we can conclude that $x = y$.

Solution: False: It is true that $\lambda x = \lambda y$, but the case $\lambda = 0$ poses a problem. For example, if $x = 0, y = 1, \lambda = 0$, then both equations are satisfied.

(c) T F If $f(x, y)$ is differentiable and attains a maximum at (a, b) in the region $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$, then $f_x(a, b) = f_y(a, b) = 0$.

Solution: False: This is true if (a, b) is in the interior of the region, but not necessarily if $|a| + |b| = 1$.

(d) T F It is possible that a function $f(x, y)$ can have no extrema along a level curve $g(x, y) = 0$.

Solution: True: for example $f(x, y) = x$ and $g(x, y) = y = 0$.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.