Discussion 10 Worksheet Answers

Global maxima and minima

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MATH 53 Multivariable Calculus

1 Finding Extrema

Find the global maxima and minima of the following functions on their indicated domains.

1. The function $f(x, y) = x^2 - y$ on the domain $D = [0, 2] \times [0, 2]$.

   Solution: On the domain, x^2 is maximized at $x = 2$ and y is minimized at $y = 0$, so f is maximized at the point $(2, 0)$ (with value 4) and similarly f is minimized at $(0, 2)$, with value -2.

2. The function $f(x, y) = x - y$ on the domain $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

   Solution: The gradient vector $\langle 1, -1 \rangle$ is never zero, so all extrema lie on the boundary. We parameterize the boundary $x = \cos(t), y = \sin(t)$, so that $f(\cos(t), \sin(t)) = \cos(t) - \sin(t)$. The derivative of this function is $-\sin(t) - \cos(t)$, which vanishes at $t = 3\pi/4$ and $t = 7\pi/4$. These correspond to minima and maxima, respectively.

3. The function $f(x, y) = x^2 - xy + y^2 - 3y$ on the region bounded by the x and y axes and the line $x + y = 4$.

   Solution: The gradient vector is $\langle 2x - y, -x + 2y + 1$, which vanishes only at the point $(1, 2)$, which is in the interior of the domain. We see $f(1, 2) = -3$. As this is the only critical point on all of $\mathbb{R}^2$ and it is a minimum (for example by the second derivative test), it is a global minimum. We examine the three boundary lines to look for possible maxima:
   On the x-axis with $0 \leq x \leq 4$, we have $f(x, 0) = x^2$, which is maximized at $f(4, 0) = 16$.
   On the y-axis, with $0 \leq y \leq 4$, we have $f(0, y) = y^2 - 3y$, which is maximized at $f(0, 4) = 4$.
   Finally, on the line $x = 4 - y$ with $0 \leq y \leq 4$, we have $f(4 - y, y) = 3y^2 - 15y + 16$, which is maximized at $f(4, 0) = 16$.
   So, altogether, there is a minimum at $f(1, 2) = -3$ and a maximum at $f(4, 0) = 16$.

1.1 Past midterm problems

1. Find the following. If an expression is undefined, say so.
   - dy/dx, where $x = 2 \sin t$, $y = 3 \cos t$. Express your answer as a function of t.
   - The area of the region between the curve whose expression in polar coordinates is $r = e^\theta$ $(0 \leq \theta \leq \pi/2)$, the line $\theta = 0$, and the line $\theta = \pi/2$.
   - $\lim_{(x,y)\to(0,0)} x/y$.
   - The equation of the plane tangent to the surface $z = (x + y)^{1/2}$ at the point where $x = 2, y = 7$.

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$$\frac{\partial^2}{\partial x \partial y}(f(x)g(y)),$$
where f and g are differentiable functions. (Express your answer in terms of f and g and their derivatives.)

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$$\int_0^1 \vec{\jmath} \times (t^2\vec{\imath} + e^{-t^2}\vec{\jmath} + (\tan t)\vec{k})dt$$
where $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ are the standard basis vectors in $\mathbb{R}^3$.

2.
- Use the cross product to find numbers p, q, r, and s such that the plane $px+qy+rz+s=0$ goes through the points $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.
- Now find a DIFFERENT set of numbers p', q', r', and s' such that the plane $p'x+q'y+r'z+s'=0$ still goes through these three points.

3. Consider the curve described in polar coordinates by $r = 2 + \cos 2\theta$.
   - Explain, without doing any computation, why the area enclosed by the curve must be less than 9π.
   - Compute the area enclosed by the curve.
   - Sketch the curve.

Solution: 1.

- We have $dx/dt = 2\cos t$ and $dy/dt = -3\sin t$, so

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{3}{2}\tan t$$

- This is just

$$\frac{1}{2}\int_0^{\pi/2} e^{2\theta} d\theta = \frac{1}{4}(e^{\pi} - 1)$$

- The limit does not exist (taking the limit along the x-axis gives 0; taking the limit along the line $x = y$ gives 1).

- We have $\partial z/\partial x = \partial z/\partial y = \frac{1}{2}(x+y)^{-1/2}$. At $(2, 7)$ these derivatives are both $1/6$, and $z = 3$, so the tangent plane equation is

$$z - 3 = \frac{1}{6}(x-2) + \frac{1}{6}(x-7).$$

- Taking the derivative with respect to x and holding y constant gives $f'(x)g(y)$. Then taking the derivative with respect to y and holding x constant gives $f'(x)g'(y)$.

- The integrand simplifies to $-t^2\vec{k} + (\tan t)\vec{\imath}$. Doing Calc 1 integration then gives the result as $-\frac{1}{3}\vec{k} - \log(\cos 1)\vec{\imath}$.

2.

- The vectors $(-1, 2, 0)$ and $(-1, 0, 3)$ both point parallel to the plane (why?). So the cross product of the first with the second, $(6, 3, 2)$ gives a vector pointing normal to the plane. This means that the equation of the plane is $6x + 3y + 2z + s = 0$ for some s. Plugging in the point $(1, 0, 0)$ shows that $s = -6$, so the plane is given by $6x + 3y + 2z - 6 = 0$.

- The easy way is just to multiply everything by some nonzero scalar. Choose this scalar to be 2 (it doesn't really matter) and we get $12x + 6y + 4z - 12 = 0$.

3.

- We have $|2 + \cos 2\theta| \leq 3$ for any θ, so the curve is contained in the circle of radius 3 centered at the origin. This containing circle has area 9π, so the curve must have smaller area (since it's not equal to the whole circle).

- We use the formula $A = \frac{1}{2}\int_0^{2\pi}(2 + \cos 2\theta)^2 d\theta = 9\pi/2$.

- This is something you can check for yourself using Desmos - though please try plotting it by hand first!

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.