Discussion 1 Worksheet Answers

Curves Defined by Parametric Equations

Date: 8/27/2021

MATH 53 Multivariable Calculus

1 Recognizing Common Curves

The following table has verbal descriptions of curves in the left column, parametric equations in the center column, and verbal descriptions in the right column. Match the verbal descriptions to the corresponding Cartesian equations and parametric equations.

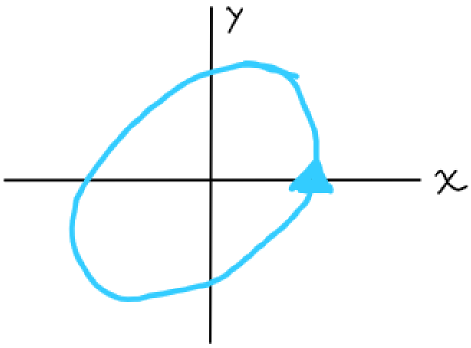
A circle of radius 1 centered at the origin	$x = 3\cos t$, $y = 2\sin t$	$y = 3x + 2$
A circle of radius 3 centered at $(2, 0)$	$x = t$, $y = 1/t$	$xy = 1$
A line of slope 3 passing through $(0, 2)$	$x = \cos t$, $y = \sin t$	$\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$
A hyperbola passing through $(1, 1)$ with the x and y axes as asymptotes	$x = t$, $y = 3t + 2$	$(x - 2)^2 + y^2 = 9$
An ellipse with semimajor axis of length 3 and semiminor axis of length 2	$x = 2 + 3\cos t$, $y = 3\sin t$	$x^2 + y^2 = 1$

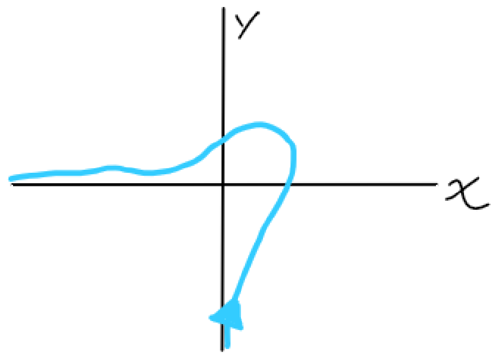
Solution: Reorganizing the table so each row corresponds to a single curve gives:

A circle of radius 1 centered at the origin	$x = \cos t$, $y = \sin t$	$x^2 + y^2 = 1$
A circle of radius 3 centered at $(2, 0)$	$x = 2 + 3\cos t$, $y = 3\sin t$	$(x - 2)^2 + y^2 = 9$
A line of slope 3 passing through $(0, 2)$	$x = t$, $y = 3t + 2$	$y = 3x + 2$
A hyperbola passing through $(1, 1)$ with the x and y axes as asymptotes	$x = t$, $y = 1/t$	$xy = 1$
An ellipse with semimajor axis of length 3 and semiminor axis of length 2	$x = 3\cos t$, $y = 2\sin t$	$\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$

2 Coordinate Graphs

Each of the curves below can be described parametrically by $x = f(t)$, $y = g(t)$ where f and g depend on the curve. The arrow in each graph below is positioned at the (x, y) value for $t = 0$. Draw graphs of each function $f(t)$, $g(t)$ separately.


y
x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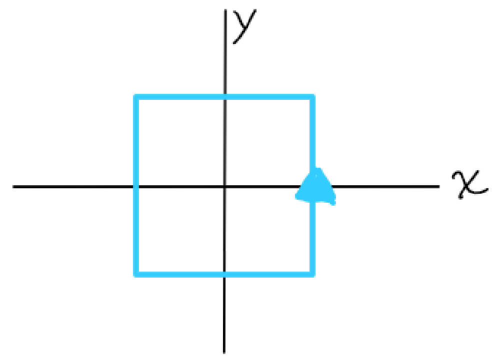
y
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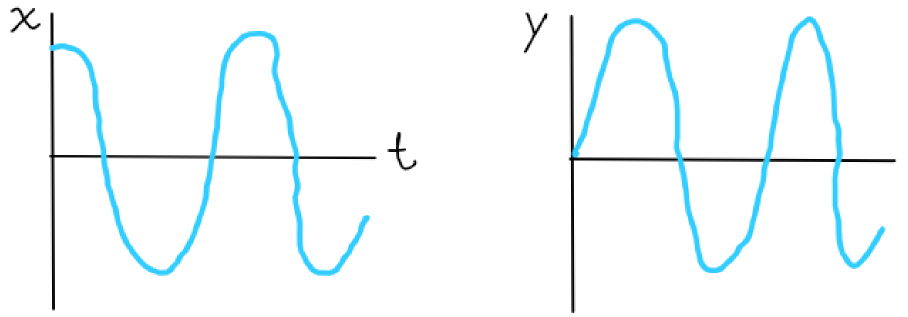
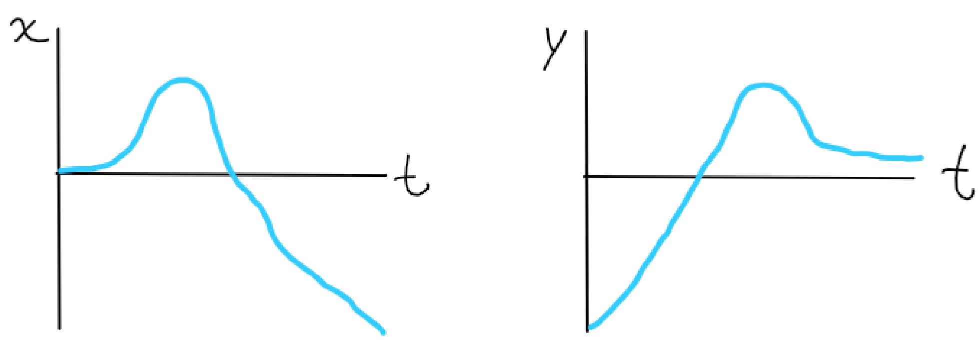
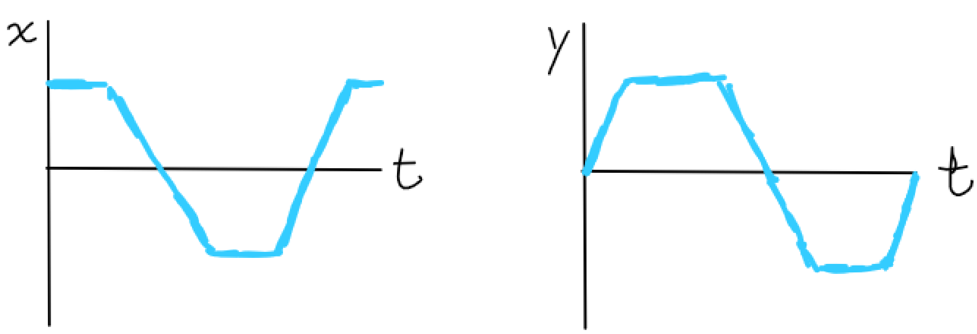
Solution: The first graphs should look like:



The x vs. t and y vs. t graphs should be "out of phase" with each other (i.e. the peaks of x should NOT occur at the same t-value as the zeros of y) - this is what causes the parametric curve to have a "tilt."
The second graphs should look like:



The (loose) symmetry of these graphs corresponds to the (loose) symmetry of the parametrized curve about the line $x = y$.
The third graphs should look like:



The points at which $f(t)$ and $g(t)$ are not differentiable correspond to the corners of the square.

3 Eliminating the Parameter

Find Cartesian equations $f(x, y) = 0$ for the following parametrized curves.

1. $x = \sqrt{t+1}$, $y = \frac{1}{t+1}$, for $t > -1$.

Solution: We have
$$x^2 = t + 1 = \frac{1}{y},$$
so the Cartesian equation is $x^2 = 1/y$. Note that the constraint $t > -1$ means that $x > 0$ and $y > 0$.

2. $x = 4 - 2t$, $y = 3 + 6t - 4t^2$.

Solution: We can solve the x equation to see $t = 2 - \frac{1}{2}x$. We then plug this in to the y equation to get
$$\begin{aligned} y &= 3 + 6\left(2 - \frac{1}{2}x\right) - 4\left(2 - \frac{1}{2}x\right)^2 \\ &= 3 + 12 - 3x - 16 + 8x - x^2 \\ &= -1 + 5x - x^2, \end{aligned}$$
which is a Cartesian equation for the curve.

3. $x = 2e^t$, $y = \cos(1 + e^{3t})$.

Solution: Note that $e^t = x/2$, so
$$y = \cos(1 + (e^t)^3) = \cos(1 + x^3/8).$$
The outer sides of this give a Cartesian equation for the curve (which holds for $x > 0$, since e^t takes on all positive values).

4 Sketching Curves

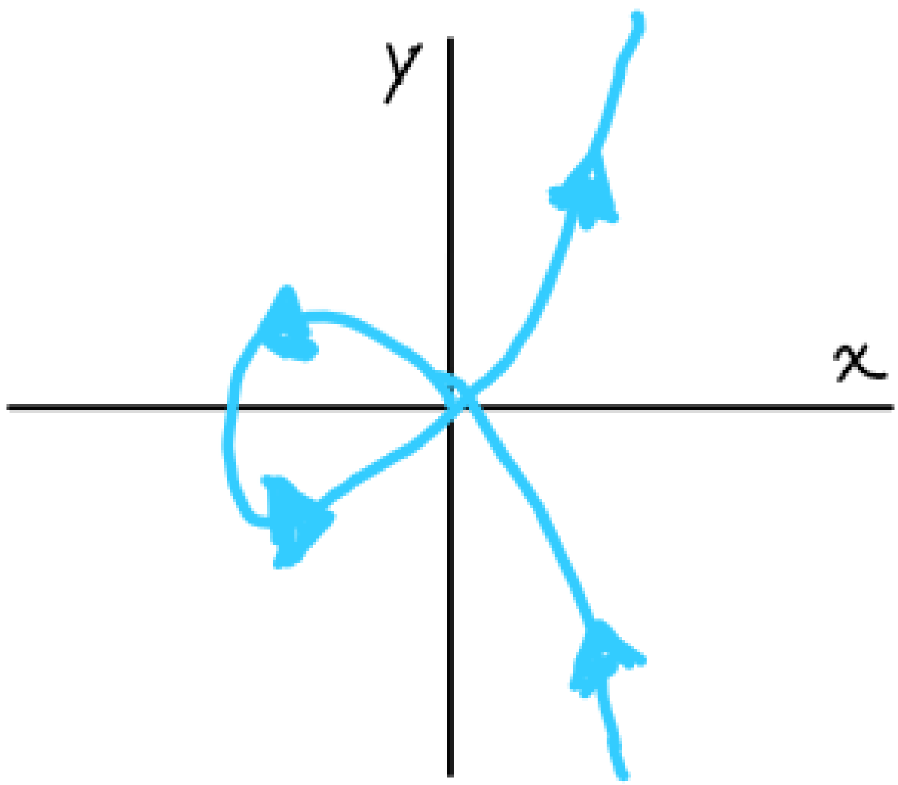
Sketch the following curves for $-\infty < t < \infty$ without using graphing calculators or similar tools.

1. $x = t^2 - 1$, $y = t^3 - t$

Solution: After plugging in sufficiently many values of t, we get a table:

t	x	y
-2	3	-6
-1.5	1.25	-1.875
-1	0	0
-0.5	-0.75	-0.375
0	-1	0
0.5	-0.75	0.375
1	0	0
1.5	1.25	1.875
2	3	6

(Some time can be saved by noting that flipping the sign of t leaves x unchanged and changes only the sign of y.) Plotting all of these points and drawing a curve through all of them in order gives our graph as:

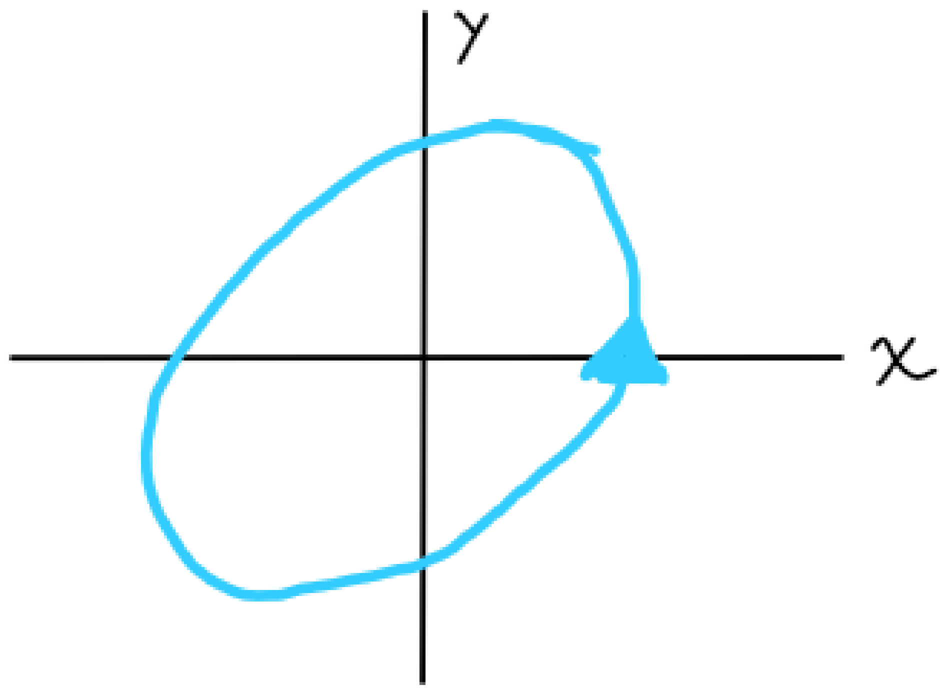


2. $x = \cos t + \frac{1}{2}\sin t$, $y = \cos t - \frac{1}{2}\sin t$

Solution: Plugging in some values of t gives a table:

t	x	y
0	1	1
$\frac{\pi}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
π	-1	-1
$\frac{3\pi}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
2π	1	1

Plotting all of these points and drawing a curve through all of them in order gives our graph as:

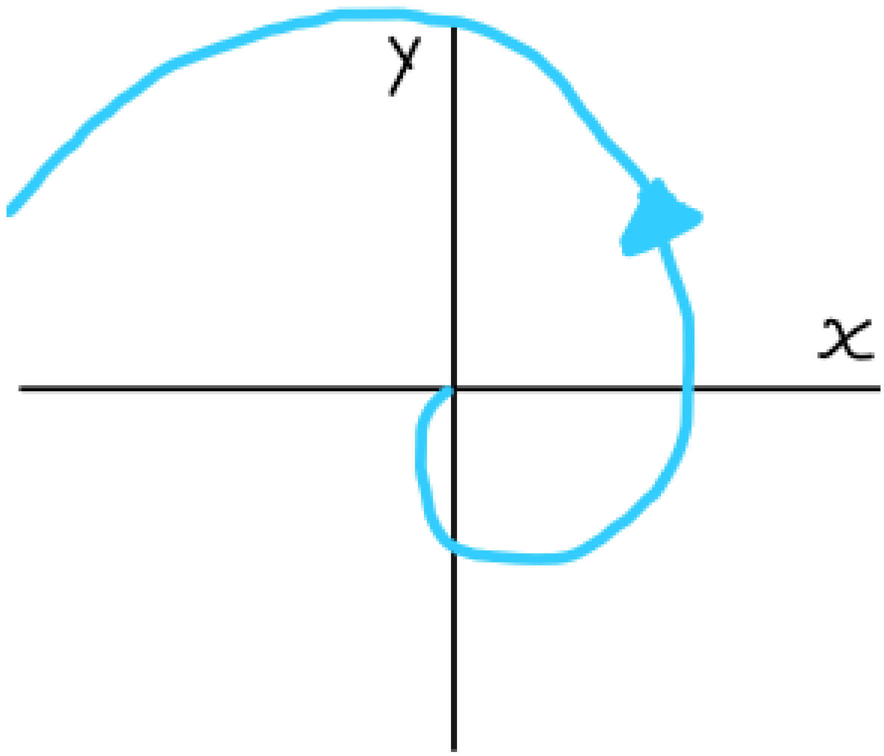


3. $x = e^t \cos t$, $y = e^t \sin t$

Solution: Again plugging in some values of t gives a table:

t	x	y
$-\pi$	-0.04	0
$-\frac{\pi}{2}$	0	-0.2
0	1	0
$\frac{\pi}{2}$	0	4.8
π	-23.14	0

Plotting some of these points and drawing a curve through all of them in order gives our graph as:



5 Intersections

Without using a graphing calculator (or similar), find all points (x, y) for which the following curves intersect.

1. The curve $x^2 - y^2 = 1$ and the curve $x = e^t - 2$, $y = e^t$

Solution: Any value of t at which the second curve meets the first must satisfy the three equations

$$\begin{aligned} x &= e^t - 2 \\ y &= e^t \\ x^2 - y^2 &= 1. \end{aligned}$$

Plugging the first two equations into the last equation gives

$$(e^t - 2)^2 - (e^t)^2 = 1.$$

Doing some algebra transforms this into $-4e^t + 4 = 1$, or equivalently $e^t = 3/4$. Plugging this back into the parametric curve equation gives

$$x = \frac{3}{4} - 2 = -\frac{5}{4}, \qquad y = \frac{3}{4}.$$

2. The curve $x^3 + y^2 + 1 = 0$ and the curve $x = t^4$, $y = t$

Solution: Any value of t at which the second curve meets the first must satisfy the three equations

$$\begin{aligned} x &= t^4 \\ y &= t \\ x^3 + y^2 + 1 &= 0. \end{aligned}$$

Plugging the first two equations into the last equation gives

$$t^{12} + t^2 + 1 = 0$$

which has no solutions (as $t^{12} + t^2$ is always nonnegative and $1 > 0$). Therefore there are no such points.

3. The curve $x = \cos t$, $y = \sin t$ and the curve $y = x^2$

Solution: The first curve is the circle $x^2 + y^2 = 1$. Therefore, to find the points of intersection, we need to find the points (x, y) satisfying $x^2 + y^2 = 1$ and $y = x^2$. Equivalently, we need to find the points (x, x^2) satisfying $x^2 + x^4 = 1$. Reorganizing, this becomes

$$(x^2)^2 + x^2 - 1 = 0,$$

so we can use the quadratic formula to find

$$x^2 = \frac{-1 \pm \sqrt{5}}{2}.$$

Only a + sign is possible here (as x^2 cannot be negative), so we see

$$(x, y) = \left(\pm\sqrt{\frac{-1 + \sqrt{5}}{2}}, \frac{-1 + \sqrt{5}}{2} \right).$$

6 Challenge

1. A *helix* is a curve shaped like a corkscrew. Parametrize a helix in $\mathbb{R}^3$ which goes through the points $(0, 0, 1)$ and $(1, 0, 1)$.

Solution: We'll come up with a helix going along the x-axis. If we look at just the y and z components of such a helix, we should get a circle passing through the point $y = 0$, $z = 1$ (since the helix is "winding around" the x-axis). We can write one such circle as a parametric curve $y = \sin 2\pi t$, $z = \cos 2\pi t$ (this is like in the xy-plane, except now our variables are z and y, and we have inserted factors of 2π to make later steps easier). So we will use the equations $y = \sin 2\pi t$, $z = \cos 2\pi t$ for our helix.
For the x-component of the helix, we want the "twists" of the helix to be evenly spaced, just like they are on a (sufficiently idealized) corkscrew. The easiest way to make this happen is to have x depend linearly on t, so that each turn around the loop (corresponding to increasing t by 1) changes x by a proportional amount. So let $x = at + b$ for some a, b to be determined. Then our curve is given by

$$x = at + b, \qquad y = \sin 2\pi t, \qquad z = \cos 2\pi t.$$

We want to choose a and b such that our curve passes through $(0, 0, 1)$ at some time (say $t = 0$) and through $(1, 0, 1)$ at some other time (say $t = 1$). Plugging these values of t into our x equation gives $a \cdot 0 + b = 0$ and $a \cdot 1 + b = 1$. We can solve these equations to get $b = 0$ and $a = 1$, so our curve is

$$x = t, \qquad y = \sin 2\pi t, \qquad z = \cos 2\pi t$$

7 True/False

Supply convincing reasoning for your answer.

(a) T F If a curve is defined by the Cartesian equation $f(x, y) = 0$, then there are no other Cartesian equations that can be used to define that curve.

> **Solution:** FALSE. For example, the unit circle could be defined by any of the equations $x^2 + y^2 - 1 = 0$, $2x^2 + 2y^2 - 2 = 0$, or $(x^2 + y^2 - 1)^2 = 0$.

(b) T F The equations $x = 2t^3$, $y = 3t^3$ give a parametrization of a line.

> **Solution:** TRUE. They parametrize the line $y = \frac{3}{2}x$. .

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.