Prepared for submission to JHEP

The conformal manifold in 2d $\mathcal{N} = (2, 2)$ SCFTs

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Abstract: The space of conformal field theories admits a Riemannian-manifold-like structure. Often, especially when there is supersymmetry involved, the manifold has more structure: complex and Kähler structures arise in well-studied examples. Simultaneously, a technique called localization has been used to compute quantities on the conformal manifold exactly (non-perturbatively). In this paper, we review these results in the two-dimensional $\mathcal{N} = (2, 2)$ case, providing more complete versions of proofs that exist in the literature.

Contents

1 Introduction

Conformal field theory (CFT) is a rich subject, and of central interest to string theory, since tree-level string theories are CFTs, and more recently because the AdS/CFT correspondence has shed deep insight into the relationship between gravity and quantum field theory. In certain cases, CFTs come to us in spaces parametrized by moduli; in these cases, the space of CFTs has the structure of a Riemannian manifold. This is particularly often the case when the CFTs under consideration have some degree of supersymmetry. This analytic structure on the space of CFTs often helps to relate correlation functions in one, well-understood

CFT to those in other CFTs. Thus, new information about CFTs stands to be gained by "zooming out" and considering the problem in a more general context.

In the present work, we review and refine the proofs and computations that have been used to compute the metric of the conformal manifold in two-dimensional, $\mathcal{N} = (2,2)$ theories. In this case, the conformal manifold is Kähler, and a new technique called localization, which can be used to compute the partition functions of supersymmetric gauged linear sigma models (GLSMs), is used to provide the exact result for the Kähler potential of the conformal manifold. Localization makes use of the fermionic symmetry of supersymmetric GLSMs to construct a nilpotent, BRST-like operator $\mathcal{Q}$. Sometimes, a large part of the action turns out to be exact under $\mathcal{Q}$, which allows for the exact computation of the partition functions and certain correlators in these GLSMs. Indeed, localization shows that the partition function is independent of the Yang-Mills coupling, which implies that the partition function is an RG-invariant. But, in [1], Ed Witten showed that two-dimensional $\mathcal{N} = (2,2)$ GLSMs flow in the infrared to non-linear sigma models with Calabi-Yau target spaces, and both flow to a CFT fixed point. Thus, localization allows us to compute the partition functions of CFTs which arise as the low-energy limits of GLSMs. We will argue, along the lines of [2], that the partition function computes the Kähler potential of the conformal manifold.

Consider a CFT p in d spatial dimensions. We can deform p by some operator $\mathcal{O}(x)$. In CFTs with a Lagrangian description, this means adding a term $\lambda \int d^d x \mathcal{O}(x)$ to the action. More generally, in theories without a Lagrangian description, this means inserting $\exp\left(\lambda \int d^d x \mathcal{O}(x)\right)$ into all correlation functions of the theory. $\mathcal{O}$ falls into one of three categories, depending on its scaling dimension Δ. If $\Delta > d$, then the operator is said to be irrelevant because the perturbed theory flows back to p in the infrared under renormalization group (RG) flow. If $\Delta < d$, then the operator is said to be relevant, because the perturbed theory flows away from p under RG flow. If $\Delta = d$, we call $\mathcal{O}$ marginal. If the perturbation by a marginal operator takes us to another CFT p', then $\mathcal{O}$ is said to be exactly marginal. The dimensionless parameter λ can then be viewed as a coordinate on the 1-dimensional space of all CFTs formed in this way. In general, there can be a set $\{\mathcal{O}_\mu\}_{\mu=1}^n$ of exactly marginal operators, and the λ^μ can be viewed as coordinates on an n-dimensional manifold of CFTs, and the $\{\mathcal{O}_\mu\}$ as a basis for the tangent space of the conformal manifold at p. Actually, this isn't quite honest: if one tries to perturbatively compute the correlation functions of the deformed theory, one will find divergences arising from the integration of a correlator with a poorly-behaved short-distance divergence. Take, for example, the two point function:

$$g_{\mu\nu} := \langle \mathcal{O}_\mu(1)\mathcal{O}_\nu(0)\rangle.$$

Conformal invariance guarantees that g_{ij} has all the properties of a Riemannian metric; $g_{\mu\nu}$ is known as the Zamolodchikov metric. The first-order change of the Zamolodchikov metric under a deformation $\exp\left(\lambda^\mu \int d^dx \mathcal{O}_\mu(x)\right)$ should be

$$\int d^dx \lambda^\rho \langle \mathcal{O}_\mu(1)\mathcal{O}_\nu(0)\mathcal{O}_\rho(x)\rangle. \tag{1.1}$$

But by conformal invariance, the correlator $\langle \mathcal{O}_\mu(1)\mathcal{O}_\nu(0)\mathcal{O}_\rho(x)\rangle$ behaves like $\frac{1}{|x|^d|x-1|^d}$, so the integral diverges logarithmically when $x \to 0$ and $x \to 1$. To make precise what we mean by a deformation then, we need to define a renormalization scheme: an assignment of physical meaning to the λ^μ. This ambiguity in the λ^μ is reminiscent of the arbitrary nature of a choice of coordinates on a manifold. Indeed, different renormalization schemes turn out to be related by changes of coordinates [3]. One can also–and this is the approach that we will take here–let a renormalization scheme define a connection on the vector bundle of operators over the conformal manifold [4, 5]. We will turn these heuristic arguments into more precise statements when we specialize to $d = 2$ in section 2.

If one restricts attention further to superconformal field theories (SCFTs) and to deformations which also preserve supersymmetry, one often discovers an additional level of structure on the conformal manifold. As mentioned above, in the $d = 2$, $\mathcal{N} = (2, 2)$ case, the conformal manifold is Kähler [2]. This extra structure on the conformal manifold imposes strong constraints on the correlation functions of operators in the class of CFTs under consideration. The constraints appear in the form of differential equations that the correlation functions must satisfy [4, 6] or in the possibility of computing exact results.

Localization has made possible the computation of exact, non-perturbative results in supersymmetric theories [7–15]. This technique has important applications to computations on the conformal manifold: in [2], it is shown that the exact partition functions computed by localization in [11] can be used to compute the Kähler potential of the 4-dimensional $\mathcal{N} = 2$ superconformal manifold. This information, combined with the four-dimensional tt^* equations discussed in [4], has been used to recursively and exactly compute all two- and three-point functions of chiral primaries in four-dimensional $\mathcal{N} = 2$ SCFTs [6]. The power of the conformal manifold formalism is thus clear: it gives us an analytic structure on the space of CFTs that allows us to compute quantities in "nearby" theories if we know them in a reference theory. Especially coupled with techniques like localization, the conformal manifold method provides wholesale insight into the structure of entire classes of CFTs.

We proceed in the following stages. In the next section, we provide an introduction to the operator formalism of CFT and to the concepts of CFT deformations, which will make relatively rigorous the concept of renormalization within the context of CFT. Moreover, we will see how and when a manifold structure can arise on the space of CFTs. We then proceed to a study of $\mathcal{N} = (2, 2)$-supersymmetric gauge theories in flat two-dimensional space and on S^2. In these theories, we will see that the computation of certain quantities (including the partition function) simplifies drastically to a one-loop calculation; moreover, the partition function turns out to be RG-invariant. In section 4, we will provide a use for the localization computation by showing that the partition function of superCFTs placed on a two-sphere computes the Kähler potential of the conformal manifold.

2 CFTs, CFT deformations, and Connections

In this section, we briefly review the operator formalism of CFTs in two dimensions, define what we mean by a CFT deformation, exhibit the manifold structure of the space of CFTs, and analyze the relationship between CFT deformations and connections on the infinite-dimensional vector bundle of operators over the conformal manifold.

2.1 CFTs in the Operator Formalism

One usually thinks of conformal field theory as defined by a set of operators (forming a representation of the Virasoro algebra) and their correlation functions [16]. Conformal invariance imposes constraints on the correlation functions: the two-point function of primary operators, for example, is proportional to some power of the distance between those operators. In two dimensions, conformal invariance dictates how correlators behave under an arbitrary holomorphic reparametrization of the complex plane, so the information contained in the correlators depends only on the complex structure of the plane with punctures at the points of insertion of the operators in consideration. We're therefore motivated, as in [17, 18], to interpret a two-dimensional conformal field theory as a rule for assigning to each Riemann surface with n punctures a set of correlators, one for each combination of n operators. In other words, a CFT is a Hilbert space $\mathcal{H}$, together with a map, for all g and n, from $\mathcal{P}(g,n)$ (the space of Riemann surfaces of genus g with n punctures and a choice of coordinates $z_1, \cdots, z_n$ vanishing at the punctures) to $\mathcal{H}^{\otimes n}$, satisfying certain properties. Note that, by the vertex-operator correspondence, an element of $\mathcal{H}^{\otimes n}$ can be seen as taking as its input n operators and outputting the correlation function of those n operators. We will discuss the additional conditions that a CFT needs to satisfy below. First, however, we digress briefly to discuss an operation on elements of $\mathcal{P} := \bigoplus_{g,n} \mathcal{P}(g,n)$ called sewing.

Let $P \in \mathcal{P}(g_1, n_1)$ (with coordinates $z_1, \cdots, z_{n_1}$) and $Q \in \mathcal{P}(g_2, n_2)$ (with coordinates $w_1, \cdots, w_{n_1}$) be two Riemann surfaces. Sewing P and Q at punctures i and j, respectively, is denoted

$$P \,{}_i\infty_j\, Q,$$

and is defined by identifying coordinate patches around the punctures in question via

$$z_i w_j = 1.$$

(If the unit disk isn't included in the coordinate patches around z_i and w_j, then we can find an isomorphic Riemann surface with punctures by rescaling z_i and w_j, as well as the other puncture coordinates, appropriately.) The Riemann surface obtained in this way depends only on the isomorphism classes of P and Q, and $P\,{}_i\infty_j\,Q \in \mathcal{P}(g_1+g_2, n_1+n_2-2)$. We can similarly define $P8^i_j$ by identifying punctures on the same Riemann surface, and $P8^i_j \in \mathcal{P}(g_1+1, n_1-2)$. Now we're in a position to give the definition of two-dimensional conformal field theory (presented more fully in [17]) that we'll use in our work. A CFT is a Hilbert space $\mathcal{H}$, furnishing a highest weight representation of the centrally-extended Virasoro algebra, and, for every element $P \in \mathcal{P}(g,n)$ (for arbitrary g,n), a ray in the space $\mathcal{H}^{\otimes n}$, a representative of which we'll write $|P\rangle$. We will sometimes write $\Sigma; z_1\cdots, z_n$ for P, where Σ is the Riemann surface associated to P and the z_i are the coordinates at the punctures. We place the following constraints on the $\{|P\rangle\}$. First, the CFT must associate tensor products to disjoint unions of Riemann surfaces. Second, $|P\rangle$ must behave under the permutation of the punctures of P by a permutation of the factors in $\mathcal{H}^{\otimes n}$. Next, let $L_n, \bar{L}_n$ be the generators of the Virasoro algebra. They satisfy:

$$\begin{aligned}
[L_n, L_m] &= L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,-m} \\
[\bar{L}_n, \bar{L}_m] &= \bar{L}_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,-m} \\
[L_n, \bar{L}_m] &= 0,
\end{aligned} \tag{2.1}$$

The stress-energy momentum tensors are given by:

$$T(z) = \sum L_n z^{n-2} \quad \bar{T}(\bar{z}) = \sum \bar{L}_n \bar{z}^{n-2}.$$

The tangent space to $\mathcal{P}(g,n)$ is spanned by n-tuples of meromorphic functions with poles only at the origin. To each such n-tuple v, we can associate an operator on $\mathcal{H}^{\otimes n}$ defined by:

$$T(v) = \frac{1}{2\pi i}\sum_i \oint T(z_i) v_i(z_i),$$

where the z_i are the coordinates vanishing at the punctures, the v_i are the components of v, and $T(z_i)$ is assumed only to act on the i-th factor in $\mathcal{H}^{\otimes n}$. We can also, completely analogously, define $\bar{T}(\bar{v})$. The second requirement in our definition is that, if we deform

P infinitesimally by v (recall that the tangent space of $\mathcal{P}(g,n)$ is spanned by the v), $|P\rangle$ changes infinitesimally by:

$$\delta\,|P\rangle = \big(T(v) + \bar{T}(\bar{v})\big)\,|P\rangle\,. \tag{2.2}$$

This is a smoothness condition: it guarantees that the CFT reflects the smooth structure of $\mathcal{P}(g,n)$. We also demand that $\delta\,|P\rangle = 0$ when P is a sphere with three or fewer punctures. This constrains the secondary-field OPEs to be determined entirely by the primary-field ones [17]. Another condition, which isn't strictly necessary, and that sometimes gets imposed is that, if we consider the Riemann sphere S with punctures at the poles and the standard coordinates z, z^{-1} there, then

$$|S; z, z^{-1}\rangle = \sum_n |n\rangle \otimes |n\rangle\,, \tag{2.3}$$

where the $|n\rangle$ are an orthonormal basis for $\mathcal{H}$. As mentioned above, this condition isn't strictly necessary. If $|S; z, z^{-1}\rangle$ didn't have this form, then conformal invariance would still require that it be a symmetric bilinear form on $\mathcal{H}\otimes\mathcal{H}$. We could therefore redefine the inner product on $\mathcal{H}$ to be the one given by $|S; z, z^{-1}\rangle$. This doesn't change any of the information in the CFT: we can compensate for this redefinition of the inner product by a change in how we extract transition amplitudes from the CFT data. Moreover, note that equation 2.3 identifies the two possible canonical ways that we have of turning a ket into a bra. However, when dealing with families of CFTs, it is easier to let the sphere state vary with the parameters of the theory space.

The final condition we place on CFTs is that they satisfy sewing relations. If $P = \Sigma; z_1, \cdots z_{n_1}$ and $Q = \Sigma'; w_1, \cdots, w_{n_2}$ are two Riemann surfaces with punctures and coordinate choices at the punctures, then the sewing relations are:

$$|P\,{}_i\infty_j\,Q\rangle = \langle S; z_i, w_j|\,(|P\rangle \otimes |Q\rangle) \tag{2.4}$$

$$|P8^i_j\rangle = \langle S; z_i, z_j|P\rangle\,. \tag{2.5}$$

where it's understood that $\langle S; z_i, w_j|$ is contracting with the i-th tensor factor in $|P\rangle$ and the j-the tensor factor in $|Q\rangle$, and similarly for the second sewing relation. Moreover, $\langle S; z_i, w_j|$ is the bra$\otimes$bra state such that $\langle S; z_i, w_j|S; w_j, z_i\rangle$ (contracting only on the inner factor) is the identity operator. In other words, sewing puncture i to puncture j corresponds to taking the trace (contracting) the i-th with the j-th tensor factors.

Equations 2.4 and 2.5 allow us to build up any $|P\rangle$ from three-punctured spheres. This is because the three-punctured sphere gives the OPE, and so the previous statement is precisely that the OPE can be used to replace arbitrary products of operators with a sum over lone operators. It's easy to see how this definition of CFT makes contact with string theory. We can think of the state $|P\rangle$ as giving the genus g contribution to the amplitude for scattering n strings: a puncture on a Riemann surface is conformally equivalent to a semi-infinite cylinder, so each puncture can be understood as propagating a string state to or from $\pm\infty$ in imaginary time. To be more precise, if we're studying n-to-m string scattering, each Riemann surface with $n+m$ punctures represents a contribution to the amplitude from a particular string interaction. If P is such a Riemann surface (hence, $|P\rangle \in \mathcal{H}^{\otimes n+m}$), we can use the inner product on $\mathcal{H}$ to create a state $|P\rangle_{in-out} \in \mathcal{H}^{\otimes n} \otimes (\mathcal{H}^*)^{\otimes m}$. $|P\rangle_{in-out}$ is the operator that effects transitions from n-string states to m-string states through the string scattering process represented by P. In the stringy interpretation, the sewing relations represent locality: the amplitude for a composite process should be the product of the amplitudes for the intermediate processes, with a sum over the possible intermediate out states of the first process contracted with the intermediate in states of the second process.

As discussed above, correlators in CFT depend only on the conformal structure of the complex plane with punctures at the operators whose correlator is being computed. The above definition of CFT should therefore give us a way to extract correlators from the state $|P\rangle$. Indeed, the operator formalism allows us to readily formulate CFTs on higher-genus surfaces. Given a Riemann surface $\Sigma; z_1, \cdots, z_n$, we can compute the correlator of the fields $\phi_1, \cdots, \phi_n$ on Σ in the following way:

$$\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle_\Sigma = \left(\bigotimes_{i=1}^{n} \langle S; z_i, w_i | \phi_i(w_i) \rangle \right) |\Sigma; z_1, \cdots, z_n\rangle \,, \tag{2.6}$$

where $|\phi_i(z_i)\rangle$ is the state corresponding to ϕ_i under the state-operator correspondence inserted at z_i and $w_i = 1/z_i$. When the sphere state is normalized as discussed above, this simply means that correlators are gotten by taking the inner-product of the state $|\Sigma; z_1, \cdots, z_n\rangle$ with the corresponding operator states. Otherwise, we have to be careful to use the sphere state, and not the inner product on $\mathcal{H}$, to turn kets into bras.

We can, in particular, extract the OPE coefficients from this information. Geometrically, this follows from the fact that if an operator ϕ_i enters the unit disk around the insertion of another operator ϕ_j, we can replace the Riemann surface Σ with the sewing product of a Riemann surface with one fewer puncture and a three-punctured sphere. The states corresponding to ϕ_i and ϕ_j are inserted into two of the punctures of the sphere, and the third puncture outputs the OPE expansion of the operators. More traditionally, we can simply compute the OPE coefficients from the three-point functions as usual.

To conclude this section, we note that the above discussion can be rephrased in more mathematical terms in the language of category theory [19] in a manner very similar to the mathematical formulation of topological quantum field theory. Let $\mathcal{C}$ denote the category whose objects are smooth, closed, oriented 1-manifolds (disjoint unions of oriented circles) and whose morphisms are Riemann surfaces whose boundary can be identified with the "in-" and "out-" 1-manifolds (with the orientation of the "out-" manifold reversed). Then it can be seen from the above discussion that a conformal field theory is a monoidal functor from $\mathcal{C}$ to some category of Hilbert spaces (whose exact properties we won't discuss here) satisfying certain smoothness properties and sewing relations. A monoidal functor in this case takes disjoint unions of circles to tensor products of Hilbert spaces. We refer the reader to [19] for further detail.

2.2 Infinitesimal CFT Deformations

Consider a CFT p with Hilbert space $\mathcal{H}$. For each $P \in \mathcal{P}$, we define an infinitesimal deformation as an assignment $\delta\,|P\rangle$ to P. The deformed ket is seen as living in $\mathcal{H}$. We take the two-punctured sphere state in p to be normalized as in 2.3, but allow for the possibility that it loses its normalization under the deformation. We note that choosing the deformed states to belong to $\mathcal{H}$ corresponds to making an arbitrary choice of identifying the Hilbert space of the deformed theory with the Hilbert space of p. In order to be an infinitesimal deformation, the transformation is required to satisfy

$$\begin{aligned}\delta \left| P \,{}_i\infty_j\, Q \right\rangle = &\left(\delta \left\langle S; z_i, w_j \right|\right) \left(\left|P\right\rangle \otimes \left|Q\right\rangle\right) \\ &+ \left\langle S; z_i, w_j \right| \left(\delta \left|P\right\rangle \otimes \left|Q\right\rangle\right) + \left\langle S; z_i, w_j \right| \left(\left|P\right\rangle \otimes \delta \left|Q\right\rangle\right). \end{aligned} \tag{2.7}$$

In other words, an infinitesimal CFT deformation preserves the sewing condition to first order. The infinitesimally deformed theory has the same inner product as $\mathcal{H}$. (As an aside, we remark that, if we had demanded the two-punctured sphere state to be normalized as in equation 2.3 above, the first term on the RHS of of equation 2.7 would be zero, but we'd have to add another term to 2.7 compensating for the fact that the deformed theory has a different inner product on its Hilbert space.)

An automorphism of $\mathcal{H}$ induces an automorphism on the $\mathcal{H}^{\otimes n}$ for all n. The infinitesimal form of such an automorphism is an endomorphism of $\mathcal{H}$ extended to all $\mathcal{H}^{\otimes n}$ by the Leibniz rule and constitutes an infinitesimal deformation. This deformation, of course, doesn't change the theory, but this fact will turn out to be important because the subtraction of counterterms in a renormalization scheme like minimal subtraction will correspond to a change of basis in $\mathcal{H}$. It will turn out that only certain choices of infinitesimal deformation will be capable of being integrated, even amongst infinitesimal deformations that define isomorphic theories. It's therefore crucial that we consider these "trivial" deformations.

If we wanted to compute higher-order contributions to a CFT deformation, we might expect that these contributions have terms like $\delta^2 \left|P\right\rangle, \delta^3 \left|P\right\rangle$, etc. in them. However, δ is only defined on the surface states $\{\left|P\right\rangle\}_{P\in\mathcal{P}}$, and $\delta \left|P\right\rangle$ might not be a surface state in the CFT p so that δ^p might not be defined on all surface states. On the other hand, any derivation Δ on the tensor algebra of $\mathcal{H}$ and its dual will preserve the sewing relations to all orders, since Δ satisfies the Leibniz rule. Δ also has the advantage of being defined on all states in $\mathcal{H}$ and tensor powers of it and its dual, which a CFT deformation as defined above doesn't necessarily do. In many cases, the CFT axioms and the $\{\delta \left|P\right\rangle\}_{P\in\mathcal{P}}$ will be enough to determine a derivation Δ that is identically δ on the surface states. In [20], the author explicitly verifies this for the case of a certain deformation on the space of toroidal compactifications. Indeed, if we have a family of CFTs that we wish to think of as a manifold, we can define a connection on that manifold and use that connection to parallel transport surface states from other CFTs to p. The Hilbert spaces of the CFTs form a bundle over the manifold, and the connection therefore defines a derivation on the tensor algebra of this bundle. Thus, in the case of a conformal manifold, we expect that all deformations arise as derivations. We will discuss this in more detail below.

We now consider a non-trivial CFT deformation. Let $\Sigma; z_1, \cdots, z_n$ be a Riemann surface with punctures. Let $\mathcal{O}$ be a marginal operator, and D_i be a disk of radius 1 centered on the i-th puncture of Σ. We can define the following infinitesimal deformation of a CFT:

$$\delta \langle \Sigma; z_1, \cdots, z_n | = \epsilon \int_{\Sigma - \cup_i D_i} d^2 z \, \langle \Sigma; z_1, \cdots, z_n, z \mid \mathcal{O}(z) \rangle \,, \tag{2.8}$$

where ϵ is infinitesimal and $\langle \Sigma; z_1, \cdots, z_n|$ is the state that we contract with $|\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$ to extract the correlator $\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle_\Sigma$ in equation 2.6. We present the deformation as a deformation on $\langle \Sigma|$ instead of $|\Sigma\rangle$ so that the deformations of the correlators are more obvious, but since the sphere state provides a non-degenerate metric on $\mathcal{H}$, it is possible to go from one to the other and back without losing any information. Moreover, by the inner product on the RHS of equation 2.8, we mean to add a puncture in Σ with local coordinate z, then take the inner product of the state corresponding to $\mathcal{O}$ with the tensor factor corresponding to z in $\langle \Sigma; z_1, \cdots, z_n, z|$. Marginality of $\mathcal{O}$ guarantees that the above integral is independent of the choice of coordinate z. It's important that the D_i are the unit disks in order for the sewing relation to be preserved, since the sewing operation joins surfaces along the unit disks around the punctures in question. However, it's easy to see that if we replace the unit disks by smaller disks, the difference between the two deformations is just a basis change. If we let $r < 1$ and δ^r denote a deformation as in equation 2.8 except with the D_i replaced by the disks D_i^r of radius r about the punctures, we find that:

$$\begin{aligned}\delta^r \langle \Sigma; z_1, \cdots, z_n| = &-\epsilon \int_{\Sigma - \cup_i D_i} d^2z \, \langle \Sigma; z_1, \cdots, z_n, z \mid \mathcal{O}(z)\rangle \\ &+ \epsilon \int_{\cup_i (D_i - D_i^r)} d^2z \, \langle \Sigma; z_1, \cdots, z_n, z \mid \mathcal{O}(z)\rangle \,. \end{aligned} \tag{2.9}$$

But we can use the OPE (which is given to us by the thrice-punctured sphere state) to replace the integral in the region near the i-th puncture in the following way: Σ with an extra puncture at coordinate z is just the product of sewing together Σ and a thrice-punctured sphere with punctures at $0, \infty$, and z, and coordinates such that the coordinate at 0 is just $1/z_i$, the coordinate near ∞ is z_i, and the coordinate near z is arbitrary. We thus have:

$$\langle \Sigma; z_1, \cdots, z_n, z| = \langle \Sigma; z_1, \cdots, z_n| \otimes \langle S; 1/z_i, z_i, z| \, |S; z_i, 1/z_i\rangle \,, \tag{2.10}$$

(with $|S; z_i, 1/z_i\rangle$ inserted to contract the i-th tensor factor of $\langle \Sigma; z_1, \cdots, z_n|$ with the first tensor factor of $\langle S; 1/z_i, z_i, z|$),so that

$$(\delta^r - \delta) \langle \Sigma| = \epsilon \langle \Sigma; z_1, \cdots, z_n| \sum_i \left(\int_{\cup_i (D_i - D_i^r)} d^2z \, |S; z_i, 1/z_i\rangle \langle S; z_i, 1/z_i, z \mid \mathcal{O}(z)\rangle \right) . \tag{2.11}$$

In other words, the difference between integrating over the unit disk and over a smaller disk is just induced by a linear transformation on $\mathcal{H}$, so that using a smaller radius also corresponds to an infinitesimal CFT deformation. Since a change in the region of integration can be compensated by a linear transformation, and conversely since any deformation by a linear transformation doesn't change the theory either, the infinitesimal CFT deformations that produce distinct CFTs are parametrized by the marginal operators $\mathcal{O}$.

We will eventually restrict attention to deformations which preserve supersymmetry. Everything we've done above carries over, except that we have to be careful that the deformation doesn't violate the symmetry. Supersymmetry can be thought of as a transformation of $\mathcal{H}$ which preserves correlation functions, i.e. preserves the states $\{|P\rangle\}_{P \in \mathcal{P}}$. Alternatively, one can generalize the preceding discussion to super-Riemann surfaces [21], and then consider only deformations of the super-Riemann surfaces. This will manifestly preserve supersymmetry.

2.3 Connections and Finite Deformations

We asserted that the space of CFTs often has a manifold structure on it. Let us see in more detail how this might come about. We have seen above that the space of infinitesimal deformations is parametrized by the marginal operators, so we have a vector space of infinitesimal deformations. Let $\{\mathcal{O}_\mu\}$ be a basis for this space. We can think of this space as the tangent space to the conformal manifold (modulo several problems that we will encounter later). We want to be able to form some sort of exponential map that can take us from infinitesimal transformations to finite ones. To this end, suppose that we did indeed have a manifold structure on the space of CFTs $\mathcal{M}$; for each $x \in \mathcal{M}$, there is a corresponding Hilbert space $\mathcal{H}_x$. $\{\mathcal{H}_x\}_{x\in\mathcal{M}}$ forms an infinite-dimensional vector bundle H over $\mathcal{M}$. We can therefore study connections on H (and the natural ones induced on its dual and tensor powers): each such connection gives a derivation on the tensor algebra of $\mathcal{H}_x$ for every $x \in \mathcal{M}$. A connection therefore gives us a CFT deformation at each point on the conformal manifold. On the other hand, infinitesimal CFT deformations do not necessarily arise as connections on H. In [20], the author explicitly exhibits the canonical deformations by marginal operators as the action of a connection in the case of CFTs arising from toroidal compactification of string theories. As mentioned above, though, this is not necessarily possible, so we will restrict attention only to deformations that do. This might restrict the class of marginal operators which we can integrate to finite deformations.

If we write the connection as $\nabla_\mu = \partial_\mu + \Gamma_\mu$, requiring that a CFT deformation arise as a connection amounts to solving for the coefficients of Γ_μ in some basis of $\mathcal{H}_x$. We have the infinite $\delta_\mu |P\rangle$, one for each $P \in \mathcal{P}$. Each such variation gives an equation linear in Γ_μ, so the system is over-determined. This gives consistency conditions (see [5], Section 5 for the beginnings of an investigation of this topic), among which is the requirement that the OPE coefficients for the expansion of a marginal operator with any other operator Φ are zero for other operators of the same dimension as Φ. In the case that Φ is marginal, this is the well-known consistency condition which guarantees that the marginals remain marginal after a deformation. Indeed, these consistency conditions give restrictions on which marginals $\mathcal{O}_\mu$ can integrate to finite deformations; we shall call *exactly marginal* those marginal operators that satisfy the consistency conditions arising from the solution of the equations for the Γ_μ, in addition to one other requirement, discussed below.

Suppose that all x in some open subset of $\mathcal{M}$ have some symmetry group, by which we mean linear transformations of $\mathcal{H}_x$ that leave the states $\{|P\rangle\}_{P\in\mathcal{P}}$ fixed. Then $\Gamma_\mu + s_\mu$, where s_μ is a one-form of symmetry transformations, gives the same action on the states $|P\rangle$ and therefore the same CFT deformation. This is, of course, no impediment to realizing an infinitesimal CFT deformation as a connection. It simply means that we won't be able to determine the coefficients of Γ_μ uniquely between states that are connected by a symmetry transformation. So, given a CFT deformation, there is some ambiguity concerning the coefficients of the Γ_μ; however, this doesn't affect the behavior of Γ_μ on the surface states.

We are interested in finite deformations, which preserve the sewing relations to all orders. These can be concocted by integrating an infinitesimal CFT deformation. We will first construct a finite CFT deformation given the assumption that the conformal manifold structure exists, and use this reasoning to derive a formula for finite deformations that can be given meaning independent of the existence of the conformal manifold. Let $x \in \mathcal{M}$, and let's restrict attention to a coordinate neighborhood U of x for which H has a local trivialization over U. This means that we can identify $\mathcal{H}_x$ for all $x \in U$. Suppose we have chosen a connection ∇_μ on H. Now, given a state $|P\rangle \in \mathcal{H}_x^{\otimes n}$, we can define a section t of $H^{\otimes n}$ which is $|P\rangle$ on some neighborhood $V \subsetneq U$ and goes to zero within U. It follows that $(\nabla_\mu t)(x) = (\Gamma_\mu t)(x) = \Gamma_\mu(x)\,|P\rangle$ for all $x \in V$. We can formally write the following transformation on the states $|P\rangle$:

$$|P\rangle_{new} = \sum_{n=0}^{\infty} \frac{\lambda^{\mu_1} \cdots \lambda^{\mu_n}}{n!} \nabla_{\mu_1} \cdots \nabla_{\mu_n} t(0). \tag{2.12}$$

We claim that this is the finite form of the infinitesimal deformation given by ∇_μ. To check that this preserves the sewing relation, note that the n-th order term in $|P\infty Q\rangle_{new}$ is

$$\frac{\lambda^{\mu_1}\cdots\lambda^{\mu_n}}{n!}\nabla_{\mu_1}\cdots\nabla_{\mu_n}\left(\langle S|\,|P\rangle\otimes|Q\rangle\right).$$

The n-th order derivative acting on the state $\langle S|\,|P\rangle\otimes|Q\rangle$ is a sum of terms, each consisting of m derivatives $\nabla_{\mu_{j_1}}\cdots\nabla_{\mu_{j_m}}$ acting on $\langle S|$, p derivatives $\nabla_{\mu_{j'_1}}\cdots\nabla_{\mu_{j'_p}}$ acting on $|P\rangle$, and $n-(m+p)$ derivatives $\nabla_{\mu_{j''_1}}\cdots\nabla_{\mu_{j''_{n-m-p}}}$ acting on $|Q\rangle$. The order of the derivatives is fixed by the requirement that $j_1<j_2<\cdots<j_m$, and similarly for the j's and the j''s. Moreover, any term with m derivatives on $\langle S|$, p derivatives on $|P\rangle$, and $n-m-p$ derivatives on $|Q\rangle$ is the same, since they just correspond to relabellings of the λ^μ. Thus, the coefficient of the $(m\mid p\mid n-m-p)$-th order term is:

$$\frac{1}{n!}\binom{n}{m}\binom{n-m}{p}=\frac{1}{n!}\frac{n!}{m!\,(n-m)!}\frac{(n-m)!}{p!\,(n-m-p)!}=\frac{1}{m!\,p!\,(n-m-p)!}.$$

On the other hand, the $(m\mid p\mid n-m-p)$-th order term in ${}_{new}\langle S|\,|P\rangle_{new}\otimes|Q\rangle_{new}$ consists of just the product of the m-th order term in ${}_{new}\langle S|$ with the p-th order term in $|P\rangle_{new}$ and the $n-m-p$-th order term in $|Q\rangle_{new}$. This evidently has the same coefficient as the corresponding term in $|P\infty Q\rangle_{new}$. Thus, at least formally, the finite deformation defined above preserves the sewing relation. We can therefore define a bona fide new CFT for each value of the λ^μ.

Now, we're being a bit fast with equation 2.12. There are a number of subtleties that need to be addressed. If the Hilbert spaces of CFTs were finite-dimensional, equation 2.12 would converge, at least in some neighborhood of $\lambda^\mu=0$. However, finite-dimensionality is not the case in the theories of interest to us, so it might not be possible to integrate the infinitesimal deformation to a finite one. There are two ways in which this can arise: either we've made a bad choice for the change of basis and radius of integration that makes it difficult to give a finite answer for the higher order deformations of the states $|P\rangle$; or, if ∇_μ corresponds to a deformation by $\mathcal{O}_\mu$ (with, possibly, an infinitesimal change of basis matrix ω_μ) and no choice of basis and radius of integration produces sensible results, the marginal operator $\lambda^\mu\mathcal{O}_\mu$ isn't exactly marginal. In fact, we *define* an exactly marginal operator to be one such that equation 2.12 gives sensible results for some choice of radius of integration and change of basis matrix, in addition to the consistency requirements discussed above. We therefore let the $\{\mathcal{O}_\mu\}$ be exactly marginal operators from here on, unless otherwise noted.

We assumed that $\mathcal{M}$ has a manifold structure and saw how we might use a connection on H to transport surface states from one CFT to another (for equation 2.12 is secretly a version of the parallel transport equation). If we do not already know that $\mathcal{M}$ is a manifold, we can still use the above discussion to construct a family of CFTs starting from a given CFT x. We just use equation 2.12 with ∇_μ the connection corresponding to deformations by an exactly marginal $\mathcal{O}_\mu$ as in equation 2.8 with, possibly, an accompanying infinitesimal change of basis ω_μ. The first-order deformation of $|P\rangle$ is therefore given by $\lambda^\mu \Gamma_\mu |P\rangle$, while the second-order term is given by:

$$\frac{1}{2}\lambda^\mu \lambda^\nu \left((\partial_\mu + \Gamma_\mu)\Gamma_\nu t\right)(0) = \frac{1}{2}\lambda^\mu \lambda^\nu \left(\partial_\mu \Gamma_\nu + \Gamma_\mu \Gamma_\nu\right)|P\rangle\,. \tag{2.13}$$

We saw above that, up to a symmetry, we can determine the coefficients of Γ_μ with respect to some basis for $\mathcal{H}$. This will determine the coefficients in terms of the CFT data: OPE coefficients, scaling dimensions of operators, etc. Since the first order changes in these values can be determined from the first-order changes in the kets $|P\rangle$, it follows that $\partial_\mu \Gamma_\nu$ is determined by the first-order deformations. Similarly, it's easy to see that the n-th order deformation of $|P\rangle$ can involve at most $n-1$ derivatives of a Γ_μ; we can therefore bootstrap our way to all the higher-order terms starting from the first-order one. Thus, we have just created a new family of CFTs parametrized by the $\mathcal{O}_\mu$, which form a basis for the space of exactly marginal operators. This gives a coordinate neighborhood of x.

We can perform a similar operation at each CFT, defining a coordinate neighborhood of every CFT. The dimension of the space of exactly marginal operators need not be the same at every CFT, in which case our attempt to create a manifold structure on the space of all CFTs fails, but often we can restrict attention to regions of theory space where this is not a problem. We won't show this here, but in the regions where the space of exactly marginal operators is constant, the transition functions between the coordinate charts as defined above are smooth. This suffices to show that there exists a manifold structure $\mathcal{M}$ on the space of CFTs. The fact that, in general, the dimension of the space of exactly marginal operators varies suggests that there might be a more general structure on the space of CFTs. It has been shown in a certain case, for example in (see [22]), that the conformal manifold is a variety. As far as I know, there isn't any literature on the subject, but it would be interesting to pursue.

2.4 Canonical Deformations and Connections

In this section, we consider several possible canonical connections on $\mathcal{M}$. Let us first establish some notation. Suppose $\{\Phi_i\}$ is a basis for $\mathcal{H}$, and that Φ_i has conformal weights $(h_i, \bar{h}_i)$ (and therefore scaling dimension $\Delta_i = h_i + \bar{h}_i$ and spin $s_i = h_i - \bar{h}_i$). We will write ${H_{\mu i}}^j$ for the coefficients of the OPE expansion of $\mathcal{O}_\mu(z)\Phi_i(0)$:

$$\mathcal{O}_\mu(z)\Phi_i(0) = \sum_j \frac{{H_{\mu i}}^j \Phi_j(0)}{z^{1+h_i-h_j}\bar{z}^{1+\bar{h}_i-\bar{h}_j}} = \sum_j \frac{{H_{\mu i}}^j \Phi_j(0)}{r^{2+\Delta_i-\Delta_j}} e^{-i\theta(s_i-s_j)}. \tag{2.14}$$

We can divide ${H_{\mu i}}^j$ into a part with integrable and non-integrable singularities:

$$\mathcal{O}_\mu(z)\Phi_i(0) = \sum_{\Delta_j \leq \Delta_i} \frac{{D_{\mu i}}^j \Phi_j(0)}{r^{2+\Delta_i-\Delta_j}} e^{-i\theta(s_i-s_j)} + \sum_{\Delta_j > \Delta_i} \frac{{F_{\mu i}}^j \Phi_j(0)}{r^{2+\Delta_i-\Delta_j}} e^{-i\theta(s_i-s_j)}, \tag{2.15}$$

where D and F stand for divergent and finite, respectively. We can extend ${D_{\mu i}}^j$ and ${F_{\mu i}}^j$ by zero to the domains where $\Delta_j > \Delta_i$ and $\Delta_j \leq \Delta_i$, respectively, and then we have

$${H_{\mu i}}^j = {D_{\mu i}}^j + {F_{\mu i}}^j. \tag{2.16}$$

If we order the basis in order of increasing scaling dimension, it follows that D is upper triangular and F only has non-zero entries below the diagonal.

Now, we can define the following three connections. First, we define $\nabla_\mu := \partial_\mu + \widehat{\Gamma}_\mu$ as follows:

$$\nabla_\mu \langle \Sigma; z_1, \cdots, z_n| = \int_{\Sigma - \cup_i D_i} d^2z \, \langle \Sigma; z_1, \cdots, z_n, z \mid \mathcal{O}_\mu(z) \rangle \, . \tag{2.17}$$

In other words, ∇_μ is basically the same as the deformation δ for marginal $\mathcal{O}_\mu$, except that

∇ is defined on all of $\mathcal{M}$, and not just at some reference CFT. In [5], it is shown that

$$\widehat{\Gamma}_{\mu i}{}^{j} = \frac{2\pi H_{\mu i}{}^{j} \delta_{s_i, s_j}}{\Delta_i - \Delta_j}. \tag{2.18}$$

In particular, $\widehat{\Gamma}_\mu$ has non-zero entries both above and below the diagonal. Thus, when computing the product $\widehat{\Gamma}_\mu \widehat{\Gamma}_\nu$ in the second-order term for the deformed surface states, the matrix product will involve the infinite sum over intermediate states and so will, in general, diverge. Presumably, this will be cancelled by the divergences in the $\partial_\mu \widehat{\Gamma}_\nu$ term, but as we will see below, a different choice of connection will be able to yield manifestly finite results.

To this end, let us define a new connection, denoted $\nabla'_\mu := \partial_\mu + c_\mu$, as follows. We define the linear operator $\omega_{\mu,\epsilon}$ for each $1 > \epsilon > 0$ by:

$$\omega_{\mu,\epsilon} \Phi_i = \sum_j \int_{D - D^\epsilon} d^2 z \frac{D_{\mu i}{}^{k} \delta_{s_i, s_j}}{r^{2 + \Delta_i - \Delta_j}} \Phi_k. \tag{2.19}$$

In other words, $\omega_{\mu,\epsilon}$ corresponds to the insertion of $\mathcal{O}_\mu(z)$ and integration over $\epsilon < |z| < 1$, but only for those operators in the operator product expansion of $\mathcal{O}_\mu(z)\Phi_i(0)$ with non-integrable singularities ($\Delta_j < \Delta_i$). We can now define ∇'_μ as:

$$\nabla'_\mu \langle \Sigma; z_1, \cdots, z_n | = \lim_{\epsilon \to 0} \left[\int_{\Sigma - \cup_i D_i^\epsilon} d^2 z \, \langle \Sigma; z_1, \cdots, z_n, z \mid \mathcal{O}_\mu(z) \rangle - \sum_{i=1}^{n} \langle \Sigma; z_1, \cdots, z | \, \omega_{\mu,\epsilon}^{(i)} \right]. \tag{2.20}$$

The connection ∇' corresponds to the following renormalization procedure. For all correlation functions on Σ, we deform them by the insertion of the operator $\mathcal{O}_\mu(z)$ and then performing a selective integration: any time $z \in D_i$, we use the OPE of $\mathcal{O}_\mu$ with the operator sitting at puncture i. This gives us a sum over operators. For those operators with non-integrable singularities in the OPE, we don't integrate inside the unit disk. For those operators with integrable singularities, we integrate over the entire disk D_i.

It can be verified that this choice of connection is upper-triangular with respect to a basis ordered by increasing scaling dimension [5]. Thus, it is a suitable choice for the computation of higher-order terms in the λ-expansion of correlation functions in a finitely-deformed theory (we will be able to compute the $c_\mu c_\nu$ term in equation 2.13). We can go one step further, though, and choose a connection whose coefficients are zero for operators of different scaling dimensions. We will call this connection $\bar{\nabla}_\mu := \partial_\mu + \bar{c}_\mu$, and it corresponds to the minimal subtraction renormalization scheme, wherein one repeats the procedure as above for ∇'_μ, except that one integrates into the unit disks up to radius ϵ. The integration from unit radius to radius ϵ produces terms that diverge as $\epsilon \to 0$, terms that go to zero as $\epsilon \to 0$, and constant terms. The renormalization scheme of ∇'_μ corresponds to keeping only the constant terms that arise. We refer the reader to [5] for details.

It should be noted that the above-mentioned connections don't necessarily preserve symmetries of a theory. When we study the manifold of superconformal field theories, we'll be interested in deformations which do indeed preserve supersymmetry. This will require a connection that is slightly modified from the ones presented in this section. The spirit of the technique is, however, very similar to the point-splitting studied here; we will present this method in section 4.

3 Localization in $\mathcal{N} = (2,2)$ Gauged Linear Sigma Models on S^2

3.1 Overview of Localization

In [1], it is shown that $(2,2)$ supersymmetric gauge theories in flat two-dimensional space (gauged linear sigma models or GLSMs, for short) flow in the infrared to Calabi-Yau nonlinear sigma models (NLSMs, for short), tree-level models of strings propagating in a Calabi-Yau background spacetime. It is believed that for every Calabi-Yau manifold with Kähler form ω, there is another Calabi-Yau manifold with the same complex structure and a Kähler form ω' cohomologous to ω for which the NLSM corresponding to this new Calabi-Yau is superconformal. The two GLSMs corresponding to the NLSMs differ by a term

$$\int d^4x d^4\theta T$$

in the action, where $-i\partial\bar{\partial}T = \omega - \omega'$. Thus, if we restrict ourselves to studying only obesrvables which are invariant under the addition of such a term, we can study properties of both NLSMs (one of which is an RG-fixed point) by studying the corresponding properties in the GLSM. This was a big development, because GLSMs are relatively easy to study, and RG-invariant quantities of interest in the infrared NLSM could be computed in the ultraviolet GLSM. We are interested in computing the partition function, which will turn out to be an RG-invariant, so a computation in the ultraviolet will yield the corresponding value in the Calabi-Yau SCFT to which the GLSM flows. We will see that we can perform an exact computation of the partition function of $\mathcal{N} = (2,2)$ GLSMs on the round two-sphere and that this computation is independent of the one dimensionful parameter in the theory: the Yang-Mills coupling. Thus, the partition function of the ultraviolet theory computes also the partition functions of the corresponding NLSM and the NLSM fixed point to which they both flow under the renormalization group.

The phenomenon by which the partition function turns out RG-invariant is part of a recent development in the computation of exact results in field theory called localization. Localization is a relatively new technique, but it's inspired by older ideas: BRST quantization and "topological twisting" ([23]). The idea is to find a fermionic symmetry $\mathcal{Q}$ such that $\mathcal{Q}^2 = 0$ (or, more generally, $\mathcal{Q}^2$ can be another symmetry of the theory). We can then modify the Lagrangian of the theory by any $\mathcal{Q}$-exact term $\int d^2z\{\mathcal{Q}, W\}$ (if $\mathcal{Q}^2 \neq 0$, then W needs to be invariant under $\mathcal{Q}^2$ up to total derivatives). By construction, this modification doesn't change the $\mathcal{Q}$-invariance of the theory. More importantly, this modification leaves the expectation values of $\mathcal{Q}$-closed observables invariant. To see this, let ϕ_α be a set of $\mathcal{Q}$-closed observables and define:

$$\left\langle \prod_\alpha \phi_\alpha \right\rangle\bigg|_t = \int [D\Phi] \exp\left(-\int d^2z \ \ (\mathcal{L} + t\{\mathcal{Q}, W\})\right) \prod_\alpha \phi_\alpha. \tag{3.1}$$

If we shift any $\phi_{\alpha'}$ by a $\mathcal{Q}$-exact term $\{\mathcal{Q}, \varphi\}$, then the correlation function remains unchanged. To see this, use the Jacobi identity (we're going to be a little reckless with the ordering of the operators for notational clarity, though the proof still goes through):

$$\delta \left\langle \prod_\alpha \phi_\alpha \right\rangle\bigg|_t = \left\langle \{\mathcal{Q}, \varphi\} \cdot \prod_{\alpha\neq\alpha'} \phi_\alpha \right\rangle\bigg|_t = \left\langle \left\{ \mathcal{Q}, \varphi \cdot \prod_{\alpha\neq\alpha'} \phi_\alpha \right\} \right\rangle\bigg|_t - \sum_\beta \langle \varphi \cdot \{\mathcal{Q}, \phi_\beta\} \cdot \prod_{\alpha\neq\alpha',\beta} \phi_\alpha \, \rangle|_t \tag{3.2}$$

We've written the action of $\mathcal{Q}$ as an anti-commutator; W needs to have fermionic statistics for the Lagrangian to be a scalar, so we do indeed take the anticommutator $\{\mathcal{Q}, W\}$. However, the ϕ_α can have either commuting or anti-commuting statistics, so we take $\{\mathcal{Q}, \phi_\alpha\}$ to mean an anti-commutator or commutator depending on whether ϕ_α is fermionic or not, respectively. We haven't been careful about minus signs coming from the super-Jacobi identity, however, since they don't affect our result. Now, since the ϕ_α are $\mathcal{Q}$-closed, each term in the sum on the RHS of the above equation is 0, and because $\mathcal{Q}$ is a symmetry, the first term on the RHS vanishes. Thus, the correlation function of any product of $\mathcal{Q}$-closed operators depends only on the $\mathcal{Q}$-cohomology classes of the operators. In particular, any correlator with at least one $\mathcal{Q}$-exact operator vanishes. It follows that the path integral 3.1 is independent of t: taking derivatives with respect to t only brings down powers of a $\mathcal{Q}$-exact term.

Since the path integral 3.1 is t-independent, we can compute the correlation functions of products of $\mathcal{Q}$-closed operators by taking the $t \to \infty$ limit. In the BRST formalism, this would correspond to actually fixing a gauge, whereas finite t corresponds to superpositions of gauge choices (Feynman gauge, for example). In any case, taking the $t \to \infty$ limit restricts the path integral to a sum (or integral) over the minima of the $\{\mathcal{Q}, V\}$ term (all other paths give zero contribution) times a one-loop determinant factor. Each stationary point is weighted by the value of the classical action $\int \mathcal{L}$ evaluated at that point. We will see the mechanics of this process in more detail in the case of interest to us later in this section. For now, however, we turn to studying $\mathcal{N} = (2,2)$ supersymmetric gauge theories on the round two-sphere.

3.2 $\mathcal{N} = (2,2)$ supersymmetric gauge theories in two dimensions

In flat space, gauge theories with $2,2$ supersymmetry can be constructed by dimensional reduction from $\mathcal{N} = 1$ theories in four dimensions [1]. The only new subtlety that arises in two dimensions is the choice of matter multiplet. In four dimensions, the matter multiplet is chosen to be in a chiral or anti-chiral representation of the SUSY algebra: $\overline{D}\Phi = 0$ or $D\Phi = 0$. In two dimensions, however, it is not inconsistent to choose $\overline{D}_+\Phi = D_-\Phi = 0$ (or the barred version of this). These are called twisted chiral superfields (or anti-twisted-chiral superfields). Chiral matter multiplets couple canonically to a twisted chiral field strength and vice versa. The two types of theories are related to each other via mirror symmetry.

Regardless of our choice of matter representation, the field content of two-dimensional GLSMs consists of the vector multiplet $(A_\mu, \sigma, \lambda, \bar{\lambda}, D)$ and the matter multiplet $(\phi, \psi, \bar{\psi}, F)$. Here, A_μ is the gauge potential, $\sigma = \sigma_1 + i\sigma_2, \phi, F$ are complex scalars, λ, $\bar{\lambda}$, ψ, and $\bar{\psi}$ are Dirac spinors, and D is a real scalar. D and F are the auxiliary scalars in the theory. The vector multiplet transforms under the adjoint representation of the gauge group G and the matter multiplet transforms under some (not necessarily irreducible) representation $\mathfrak{R}$ of G. The flat-space theory is invariant under the full $\mathcal{N} = (2,2)$ superconformal algebra. However, when the theory is coupled to the two-sphere, it's not possible to preserve both R symmetries; at most, one $SU(2 \mid 1)$ sub-algebra is preserved. If the R-symmetry preserved is R, then $SU(2 \mid 1)_A$ is preserved; otherwise, $SU(2 \mid 1)_B$ is preserved. There are, however, two different ways to construct theories with $SU(2 \mid 1)_A$ symmetry, depending on whether the matter multiplet used is chiral or twisted chiral. Mirror symmetry exchanges chiral and twisted chiral multiplets, R and $\mathcal{A}$, $SU(2 \mid 1)_A$ and $SU(2 \mid 1)_B$. Thus, a theory formulated with a vector R-symmetry R and chiral multiplets is isomorphic to one formulated with an axial R-symmetry and twisted chiral multiplets. In [8, 9], the partition function is computed in theories with chiral multiplets and a vectorlike R-symmetry. Here, we will present the computation of the partition function of theories with twisted chiral multiplets and a vector-like R-symmetry. We will follow the approach of [10], though portions of the following were developed independently by the author.

The Lagrangian of the $SU(2 \mid 1)_B$ invariant theory on the two-sphere has action $S =$

$\int_{S_2} d^2x\sqrt{h}\mathcal{L}$, where $\mathcal{L}$ is the sum of four terms:

$$\mathcal{L}_{v.m.} = \frac{1}{2g^2}\mathrm{Tr}\Big\{(F_{12})^2 + D^2 + D_\mu\sigma D^\mu\bar{\sigma} + \frac{1}{4}[\sigma,\bar{\sigma}]^2 + i\bar{\lambda}\gamma^\mu D_\mu\lambda$$
$$-\frac{i}{2}\lambda[\sigma,\gamma_3\lambda] + \frac{i}{2}\bar{\lambda}[\bar{\sigma},\gamma_3\bar{\lambda}] + \frac{\bar{\lambda}\lambda}{r}\Big\} \tag{3.3}$$
$$\mathcal{L}_{t.c.m.} = \bar{F}F + i\bar{\psi}\gamma^\mu D_\mu\psi + D_\mu\bar{\phi}D^\mu\phi - i\bar{\phi}D\phi + i\bar{\psi}\sigma\psi - \bar{\psi}\gamma_3\sigma_2\psi$$
$$+\bar{\phi}(\sigma_1^2+\sigma_2^2)\phi + i\bar{\psi}\gamma_3(P_-\bar{\lambda}+P_+\lambda)\phi - i\bar{\phi}(P_-\lambda+P_+\bar{\lambda})\gamma_3\psi \tag{3.4}$$
$$\mathcal{L}_W = W'(\phi)(iF) - \frac{1}{2}W''(\phi)\psi\psi + \frac{W(\phi)}{r} + c.c. \tag{3.5}$$
$$\mathcal{L}_{FI} = \mathrm{Tr}\left(-i\xi D + \frac{i\theta}{2\pi}F_{12}\right) \tag{3.6}$$

Here, r is the radius of S^2, g is the gauge-coupling, F_{12} is the curvature of the connection A_μ ($F_{12} = \partial_1 A_2 - \partial_2 A_1 - i[A_1, A_2]$), D_μ is the gauge- and diffeomorphism-covariant derivative, $P_\pm$ is defined as in appendix A, W is an arbitrary, gauge-invariant, holomorphic function, and ξ and θ are the Fayet-Iliopoulos and topological parameters. There is a Fayet-Iliopoulos and topological term for every $U(1)$ factor in G; we will refer to both collectively as the Fayet-Iliopoulos term, unless otherwise stated. The vector R-symmetry R acts axially on ψ and vectorially on λ: $\psi \mapsto e^{i\alpha\gamma_3}\psi$, $\lambda \mapsto e^{i\alpha}\lambda$, $\sigma \mapsto e^{-2i\alpha}\sigma$. The axial R-symmetry $\mathcal{A}$ would act oppositely on ψ and λ, but the $\frac{\bar{\lambda}\lambda}{r}$ term clearly violates such a symmetry; this is a consequence of the fact mentioned above that it is impossible to preserve both R-symmetries on the two-sphere.

Part of the difficulty of putting the GLSM on the two-sphere is that, in principle, if we could find a covariantly constant spinor $\nabla_\mu\epsilon = 0$ on S^2, then we could just promote the flat-space Lagrangian and supersymmetry variations to diff-covariant versions with SUSY parameter ϵ by the minimal prescription scheme and this would guarantee the SUSY-invariance of the S^2 theory. Such a covariantly constant spinor does not exist, however, so we must make do with conformally invariant Killing spinors (see section A.2 for explicit realizations of ϵ and $\bar{\epsilon}$):

$$\nabla_\mu\epsilon = \frac{i\gamma_\mu\epsilon}{2r}, \qquad \nabla_\mu\bar{\epsilon} = \frac{i\gamma_\mu\bar{\epsilon}}{2r}. \tag{3.7}$$

This is why $1/r$ corrections appear in the Lagrangian and the SUSY variations (see below).

In principle, one could formulate a supersymmetric theory on curved space by formulating the theory as a supergravity theory and then taking the Planck mass $M_p \to \infty$. This freezes out the gravitational degrees of freedom; we refer the reader to [26] for details. In practice, however, it is usually easier to just compute the corrections to the flat-space Lagrangian order by order in $1/r$. This is what I did. Given that we have replaced the constant SUSY parameters with Killing spinors, it can be verified that the above action is invariant under the following SUSY variations for the vector multiplet

$$\begin{aligned}
\delta\sigma &= \bar{\epsilon}\bar{\lambda}, \qquad \delta\bar{\sigma} = -\epsilon\lambda, \qquad \delta A_\mu = \frac{i}{2}\left(\bar{\epsilon}\gamma_\mu\gamma_3 + \epsilon\gamma_\mu\gamma_3\bar{\lambda}\right) \\
\delta\lambda &= \left(D - iF_{12} - \frac{i}{2}[\sigma,\bar{\sigma}]\gamma_3\right)\epsilon - iD_\mu\bar{\sigma}\gamma^\mu\bar{\epsilon} + \frac{\bar{\sigma}}{r}\bar{\epsilon} \\
\delta\bar{\lambda} &= \left(-(D + iF_{12}) + \frac{i}{2}[\sigma,\bar{\sigma}]\gamma_3\right)\bar{\epsilon} + iD_\mu\sigma\gamma^\mu\epsilon - \frac{\bar{\sigma}}{r}\epsilon \\
\delta D &= -\frac{i}{2}D_\mu\left(\epsilon\gamma^\mu\bar{\lambda} - \bar{\epsilon}\gamma^\mu\lambda\right) - \frac{i}{2}[\bar{\epsilon}\gamma_3\bar{\lambda},\bar{\sigma}] - \frac{i}{2}[\epsilon\gamma_3\lambda,\sigma]
\end{aligned} \tag{3.8}$$

and the following SUSY variations for the twisted chiral multiplet.

$$\begin{aligned}
\delta\phi &= \left(P_-\epsilon + P_+\bar{\epsilon}\right)\psi \qquad \delta\bar{\phi} = -\left(P_-\bar{\epsilon} + P_+\epsilon\right)\bar{\psi} \\
\delta\psi &= \left(P_-\epsilon + P_+\bar{\epsilon}\right)(iF) - i\sigma_1\phi\left(P_-\bar{\epsilon} + P_+\epsilon\right) - \sigma_2\phi\left(P_-\bar{\epsilon} - P_+\epsilon\right) + i\gamma^\mu\left(P_-\bar{\epsilon} + P_+\epsilon\right)D_\mu\phi \\
\delta\bar{\psi} &= \left(P_-\bar{\epsilon} + P_+\epsilon\right)(i\bar{F}) + i\bar{\phi}\sigma_1\left(P_-\epsilon + P_+\bar{\epsilon}\right) - \bar{\phi}\sigma_2\left(P_-\epsilon - P_+\bar{\epsilon}\right) - i\gamma^\mu\left(P_-\epsilon + P_+\bar{\epsilon}\right)D_\mu\bar{\phi} \\
\delta F &= \left(P_-\bar{\epsilon} + P_+\epsilon\right)\sigma_1\psi + i(P_+\epsilon - P_-\bar{\epsilon})\sigma_2\psi + (P_-\bar{\epsilon} + P_+\epsilon)\gamma^\mu D_\mu\psi - (\bar{\epsilon}P_+\lambda - \epsilon P_-\bar{\lambda})\phi \\
\delta\bar{F} &= \left(P_-\epsilon + P_+\bar{\epsilon}\right)\bar{\psi}\sigma_1 - i(P_+\bar{\epsilon} - P_-\epsilon)\bar{\psi}\sigma_2 + (P_-\epsilon + P_+\bar{\epsilon})\gamma^\mu D_\mu\bar{\psi} - \bar{\phi}(\epsilon P_+\bar{\lambda} - \bar{\epsilon}P_-\lambda)
\end{aligned} \tag{3.9}$$

We can write $\delta = \delta_\epsilon + \delta_{\bar{\epsilon}}$ and compute $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ acting on the fields to see whether δ_ϵ and $\delta_{\bar{\epsilon}}$ really realize the algebra $SU(2 \mid 1)$ on the fields. Doing this reveals (letting $\alpha := -i\bar{\epsilon}\epsilon/2r$

and $\xi^\mu = i\bar{\epsilon}\gamma^\mu\epsilon$):

$$\begin{aligned}
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\sigma &= \xi^\mu D_\mu \sigma - 2i\alpha\sigma \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\sigma} &= \xi^\mu D_\mu \bar{\sigma} + 2i\alpha\bar{\sigma} \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\lambda &= \xi^\mu D_\mu \lambda + \frac{1}{4}\nabla_\mu \xi_\nu \gamma^{\mu\nu}\lambda + i\alpha\lambda \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\lambda} &= \xi^\mu D_\mu \bar{\lambda} + \frac{1}{4}\nabla_\mu \xi_\nu \gamma^{\mu\nu}\bar{\lambda} - i\alpha\bar{\lambda} \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]D &= i\xi^\mu D_\mu D \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]A_\mu &= \xi^\rho F_{\rho\mu}
\end{aligned} \tag{3.10}$$

for the gauge multiplet and

$$\begin{aligned}
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\phi &= \xi^\mu D_\mu \phi \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\phi} &= \xi^\mu D_\mu \bar{\phi} \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\psi &= \xi^\mu D_\mu \psi + \frac{1}{4}\nabla_\mu \xi_\nu \gamma^{\mu\nu}\psi + i\alpha\gamma_3\psi \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\psi} &= \xi^\mu D_\mu \bar{\psi} + \frac{1}{4}\nabla_\mu \xi_\nu \gamma^{\mu\nu}\bar{\psi} - i\alpha\gamma_3\psi \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]F &= \xi^\mu D_\mu F \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{F} &= \xi^\mu D_\mu \bar{F}
\end{aligned} \tag{3.11}$$

for the matter multiplet. Thus, $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ acts by an isometry with parameter ξ^μ and an R-symmetry with parameter α. This is consistent with the commutation relation

$$\{\tilde{S}_\alpha, \tilde{Q}_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta}\mathcal{A}$$

in equations B.3. Furthermore, it can be verified that $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]$ is a gauge transformation with parameter $\epsilon_2\gamma_3\epsilon_1\sigma$, and similarly, $[\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}]$ is a gauge transformation with gauge parameter $-\bar{\epsilon}_2\gamma_3\bar{\epsilon}_1\bar{\sigma}$.

Finally, we note that the gauge kinetic and matter kinetic actions are δ-exact:

$$\begin{aligned}
\bar{\epsilon}\epsilon \int d^2x\sqrt{h}\ \mathcal{L}_{v.m.} &= \delta_{\bar{\epsilon}}\delta_\epsilon \int d^2x\sqrt{h}\ \mathrm{Tr}\left(-\bar{\lambda}\lambda + \frac{\bar{\sigma}\sigma}{r}\right) \\
\bar{\epsilon}\epsilon \int d^2x\sqrt{h}\ \mathcal{L}_{t.c.m.} &= \delta_{\bar{\epsilon}}\delta_\epsilon \int d^2x\sqrt{h}\ \left(i(\bar{\phi}F - \bar{F}\phi) + \frac{\bar{\phi}\phi}{r}\right).
\end{aligned} \tag{3.12}$$

$\bar{\epsilon}\epsilon$ can be easily verified to be covariantly constant, i.e. a constant, thanks to equation 3.7; moreover, since $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ is a symmetry of the theory, it follows that equations 3.12 are still true with δ_ϵ and $\delta_{\bar{\epsilon}}$ switched. Moreover, when we choose which supercharge to use in localization, $(2,2)$ supersymmetry will give us several choices. Equations 3.12, along with the fact that all commutators of δ_ϵ and $\delta_{\bar{\epsilon}}$ with themselves and each other are symmetries, tell us that whatever $\mathcal{Q}$ we choose for localization, the kinetic part of the GLSM action is $\mathcal{Q}$-exact. In particular, this means that the partition function of the GLSM is independent of g^2, the Yang-Mills coupling of the theory. g^2 is the only dimensionful parameter in the theory; it follows that, as noted above, the partition function is RG invariant. Moreover, once we specify the localizing supercharge, we will find that the Fayet-Iliopoulos action is also $\mathcal{Q}$-exact; the partition function therefore depends only the twisted superpotential W.

Let us see in more detail why the Fayet-Iliopoulos action is exact; to do this, we construct our localizing supercharge $\mathcal{Q}$. Write $\delta_\epsilon = \epsilon^\alpha Q_\alpha$ and $\delta_{\bar{\epsilon}} = \bar{\epsilon}^\alpha Q_\alpha^\dagger$. This defines the operators Q and $Q^\dagger$. Now, let ε be a commuting spinor given explicitly by

$$\varepsilon = e^{-i\frac{\phi}{2}} \begin{pmatrix} \sin\theta/2 \\ -i\cos\theta/2 \end{pmatrix}. \tag{3.13}$$

This is just one of the generating elements of the two-dimensional space of conformal spinors satisfying 3.7. Let $\bar{\varepsilon} = C\varepsilon^*$; then it can be verified that:

$$\begin{aligned} \bar{\varepsilon}\varepsilon &= -\varepsilon^\dagger C^2 \varepsilon = \varepsilon^\dagger \varepsilon = 1 \\ \varepsilon\varepsilon &= \bar{\varepsilon}\bar{\varepsilon} = 0. \end{aligned} \tag{3.14}$$

The second of relations 3.14 follows from the commuting nature of ε. Now that all of these definitions are in place, we let $\mathcal{Q} = \varepsilon^\alpha Q_\alpha + \bar{\varepsilon}^\alpha Q_\alpha^\dagger$. This will be our localizing supercharge; because ε is commuting, $\mathcal{Q}$ is anti-commuting, as expected. Moreover, it's easy to see that $\mathcal{Q}^2$ has only terms with $\{Q_\alpha^{(\dagger)}, Q_\beta^{(\dagger)}\}$; but these are symmetries of the theory by virtue of the discussion above about $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$, $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]$, and $[\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}]$. Thus, $\mathcal{Q}$ is as nilpotent as we need it to be from the discussion in 3.1. As mentioned above, 3.12 guarantees that the gauge and twisted chiral kinetic actions are $\mathcal{Q}$-exact. Moreover, it's easy to check that, up to total derivative terms $\bar{\varepsilon}\gamma^\mu\varepsilon D_\mu\sigma$ and $\bar{\varepsilon}\gamma^\mu\varepsilon D_\mu\bar{\sigma}$, D and F_{12} are $\mathcal{Q}$-exact:

$$\begin{aligned}\int d^2x D &= \int d^2x \mathcal{Q}\left(\bar{\varepsilon}\lambda - \bar{\lambda}\varepsilon\right) \\ \int d^2x F_{12} &= i\int d^2x \mathcal{Q}\left(\bar{\varepsilon}\lambda + \bar{\lambda}\varepsilon\right).\end{aligned} \tag{3.15}$$

Another way to understand the $\mathcal{Q}$-exactness of the Fayet-Iliopoulos and topological action is to note that the Lagrangian $\mathcal{L}_{F.I.}$ is a special case of a more general type of term that can be added to the action, a superpotential term. The superpotential action is the the top component of a chiral multiplet field whose R-charge is 2. For R-charge 2, the supersymmetry variation of the spinor in a chiral multiplet is, up to a total derivative, the top component of the multiplet (see, for example, [9]). Thus, the integrated top component of a chiral multiplet is exact. But the field strength multiplet is exactly such a multiplet, and it can be shown that $D + iF_{12}$ is the top component of this multiplet. Thus, the F.I. action is $\mathcal{Q}$-exact.

Since $\mathcal{L}_{F.I.}$, $\mathcal{L}_{t.c.m}$, and $\mathcal{L}_{v.m.}$ are all $\mathcal{Q}$-exact, the partition function depends only on the parameters of the twisted super-potential. However, we can compute the partition function with any linear combination of three contributions to the action. We will choose to use $t(\mathcal{L}_{t.c.m.} + \mathcal{L}_{v.m.} + \frac{i}{2}\mathrm{Tr}(D)) + \mathcal{L}_W$ as the action with which we localize and take the $t \to \infty$ limit. In this limit, when we compute the path integral computing the partition function, we will find that points in field space that are not minima of $\mathcal{L}_{t.c.m.} + \mathcal{L}_{v.m.} + \frac{i}{2}\mathrm{Tr}(D)$ have vanishingly small contributions to the path integral. Thus, the path integral localizes to the minima (we will denote these collectively by φ_0) of $\mathcal{L}_{t.c.m.} + \mathcal{L}_{v.m.} + \frac{i}{2}\mathrm{Tr}(D)$. Writing the expansion $\varphi = \varphi_0 + \frac{1}{\sqrt{t}}\tilde{\varphi}$ for the fields of the theory and expanding the action around the fixed points φ_0, we discover that the path integral simply reduces to a sum (or integral) over the minima of the localizing action, weighted by both $\exp\{-\int d^2x\sqrt{h}\ \mathcal{L}_W\}$ and a 1-loop determinant arising from the quadratic term in the expansion of the action around the fixed points. (The $t \to \infty$ limit is formally the same as the $\hbar \to 0$ limit.) Moreover, as shown in [23], whenever a theory has a fermionic symmetry $\mathcal{Q}$, the path integral defining the expectation value of $\mathcal{Q}$-invariant observables reduces to an integral over fixed points of $\mathcal{Q}$ times a one-loop determinant. Thus, we only have to examine field configurations that are fixed points both of $\mathcal{Q}$ and the localizing Lagrangian. We will perform this computation in a moment. First, however, we have to address the nuance of gauge-fixing, to which we turn in the next subsection.

3.3 Gauge-Fixing and $\mathcal{Q}_{BRST}$

If we attempted to compute the path integral using just the action $\int d^2x\sqrt{h}\ \mathcal{L}$, we would encounter divergences arising from the existence of zero modes. These zero modes arise because of the large gauge redundancy in our description of the physics. In the present case, the problem can be addressed by a slight modification of the normal BRST gauge-fixing scheme. We will encounter some difficulties because we'd like for the localizing supercharge $\mathcal{Q}$ and the BRST supercharge to interact in a way that guarantees that the localization argument still carries through in a convenient way. We will address these difficulties in a moment; let us first, however, review the standard BRST method for gauge fixing.

In the BRST method, one introduces the operator $\mathcal{Q}_{BRST}$, which is anticommuting and acts as an infinitesimal gauge transformation with fermionic parameter/field c on the physical fields of the theory. It follows that any gauge-invariant action is annihilated by $\mathcal{Q}_{BRST}$. One introduces moreover the Grassmann-odd field $\bar{c}$ and the Grassmann-even field b, and declares the following BRST variations of the "ghost" fields $c, \bar{c}, b$ (all three of which transform in the adjoint representation, i.e. $c = c^A t_A$, where the t_A are infinitesimal Hermitian generators of G):

$$\begin{aligned} &\mathcal{Q}_{BRST}c = icc = ic^A c^B t_A t_B = \frac{i}{2}c^A c^B [t_A, t_B] \\ &\mathcal{Q}_{BRST}\bar{c} = b, \quad \mathcal{Q}_{BRST}b = 0. \end{aligned} \tag{3.16}$$

It can be verified that $\mathcal{Q}_{BRST}^2 = 0$ on all fields. Thus, by the same arguments as we saw above, the modification of the action by any $\mathcal{Q}_{BRST}$-exact term doesn't change the result. We choose to modify the action by a term

$$\begin{aligned} S_{g.f.} &= \mathcal{Q}_{BRST}\int d^2x\sqrt{h}\ \mathrm{Tr}\left(\bar{c}\left(\mathcal{G} - \frac{1}{2}b\right)\right) \\ &= \int d^2x\sqrt{h}\ \mathrm{Tr}\left(b\mathcal{G} - \bar{c}\left(\delta_{gauge}(c)\mathcal{G}\right) - \frac{1}{2}b^2\right), \end{aligned} \tag{3.17}$$

where $\mathcal{G}$ is some gauge-fixing functional in the adjoint representation. Different choices of $\mathcal{G}$ correspond to different gauge fixing choices. We note that b appears at most quadratically in the action $S_{g.f.}$, so we can replace it with its equation of motion, $b = \mathcal{G}$. The result is:

$$S_{g.f.} = \int d^2x \sqrt{h} \, \mathrm{Tr} \left(\frac{1}{2} \mathcal{G}^2 - \bar{c} \left(\delta_{gauge}(c) \mathcal{G} \right) \right). \tag{3.18}$$

We note that $\mathcal{G} = \nabla_\mu A^\mu$ is the normal choice for the gauge-fixing functional, but we will be interested in a slightly different gauge choice later. For now, we leave $\mathcal{G}$ unspecified.

The nuance that arises in the present case is that the action that we need to use for the path integral isn't just the action of equations 3.3-3.6; we need to actually add $S_{g.f.}$ to the action. With the action thus modified, it is no longer guaranteed that the total action is $\mathcal{Q}$-exact. We need to find some other fermionic symmetry operator that combines both $\mathcal{Q}$ and $\mathcal{Q}_{BRST}$ to carry through the localization argument. In order to do this, we need to generalize the discussion of $\mathcal{Q}_{BRST}$ from above. Instead of $\mathcal{Q}_{BRST}^2 = 0$, we let $\mathcal{Q}_{BRST}^2$ be a gauge transformation by some parameter a_0. This is effected by taking:

$$\begin{gathered} \mathcal{Q}_{BRST}(\text{non-ghost fields}) = \delta_{gauge}(c)(\text{non-ghost fields}) \quad \mathcal{Q}_{BRST} c = a_0 + icc \\ \mathcal{Q}_{BRST} a_0 = -(\mathcal{Q} + \mathcal{Q}_{BRST})\Delta \quad \mathcal{Q}_{BRST} \bar{c} = b \quad \mathcal{Q}_{BRST} b = i[a_0, \bar{c}], \end{gathered} \tag{3.19}$$

where Δ is the gauge parameter that appears in the action of $\mathcal{Q}^2$ on the physical fields. Moreover, we take the following supersymmetry variations of the ghost fields:

$$\mathcal{Q} c = \mathcal{Q} a_0 = \mathcal{Q} \bar{c} = 0 \quad \mathcal{Q} b = (\xi^\mu \nabla_\mu + \delta_{gauge}(\Delta)) \bar{c}. \tag{3.20}$$

Now, we can compute:

$$\begin{gathered} \mathcal{Q}_{BRST}^2(\text{non-ghost fields}) = \delta_{gauge}(a_0)(\text{non-ghost fields}) \\ \mathcal{Q}_{BRST}^2 c = i[a_0, c] \quad \mathcal{Q}_{BRST}^2 \bar{c} = i[a_0, \bar{c}] \\ \mathcal{Q}_{BRST}^2 a_0 = -\mathcal{Q}_{BRST}(\mathcal{Q}_{BRST} + \mathcal{Q})\Delta \\ \mathcal{Q}_{BRST}^2 b = i[a_0, b] + i\{\mathcal{Q}_{BRST} a_0, \bar{c}\}, \end{gathered} \tag{3.21}$$

$$\begin{aligned}
\mathcal{Q}^2\,(\text{non-ghost fields}) &= (L_\xi + R + \delta_{gauge}(\Delta))\,(\text{non-ghost fields})\\
\mathcal{Q}^2 c = \mathcal{Q}^2 a_0 &= \mathcal{Q}^2 \bar{c} = 0 \\
\mathcal{Q}^2 b &= i\{\mathcal{Q}\Delta, \bar{c}\},
\end{aligned} \tag{3.22}$$

where L_ξ is a Lie derivative in the direction of ξ^μ, the Killing vector defined above, and R is the R-symmetry which appears in $\mathcal{Q}^2$. Finally, we compute

$$\begin{aligned}
&\{\mathcal{Q}_{BRST}, \mathcal{Q}\}(\text{non-ghost fields}) = 0 \\
&\{\mathcal{Q}_{BRST}, \mathcal{Q}\}\bar{c} = (L_\xi + \delta_{gauge}(\Delta))\,\bar{c} \\
&\{\mathcal{Q}_{BRST}, \mathcal{Q}\}b = (L_\xi + \delta_{gauge}(\Delta))b + i\{\mathcal{Q}_{BRST}\Delta, \bar{c}\}.
\end{aligned} \tag{3.23}$$

We have not computed $\{\mathcal{Q}_{BRST}, \mathcal{Q}\}$ on c, a_0 because we will take our action to be $\hat{\mathcal{Q}}V := (\mathcal{Q} + \mathcal{Q}_{BRST})V$, where V does not contain any terms with a_0 or c, and since all we need for the localization argument to carry through is that $\hat{\mathcal{Q}}^2 V = 0$, $\hat{\mathcal{Q}}^2 a_0$ and $\hat{\mathcal{Q}}^2 c$ are irrelevant for our purposes. It follows from the three sets of equations above that, if we take the R-charges of $\bar{c}$ and b to be zero, then $\hat{\mathcal{Q}}^2$ acts as a combination of isometry in the direction of ξ^μ, an R-symmetry as before, and a gauge transformation with parameter $\Delta + a_0$. Moreover, since the $\mathcal{Q}$-exact parts of the non-gauge-fixed action were $\mathcal{Q}$ of a gauge-invariant quantity, it follows that those same terms are also $\hat{\mathcal{Q}}$-exact. Now we take the gauge-fixing action to be:

$$S_{g.f.} = \hat{\mathcal{Q}} \int d^2x \sqrt{h}\ \mathrm{Tr}\left(\bar{c}\left(\mathcal{G} - \frac{1}{2}b\right)\right) = \int d^2x \sqrt{h}\ \mathcal{L}_{gh}. \tag{3.24}$$

This is actually different from the gauge-fixing action described above, since we have both a term arising from $\mathcal{Q}$ acting on the Lagrangian and since $\mathcal{Q}_{BRST}$ acts differently from the operator as it was originally presented. The terms arising from the action of $\mathcal{Q}$ are $-\mathrm{Tr}(\bar{c}\mathcal{Q}\mathcal{G})$ and $\frac{1}{2}\mathrm{Tr}(\bar{c}\mathcal{Q}b)$. Both of these terms can be absorbed into the definition of c. The term arising from the slight difference in our definition of $\mathcal{Q}_{BRST}$ is $\frac{i}{2}\mathrm{Tr}(\bar{c}[a_0, \bar{c}])$, and cannot be ignored, though it won't present any difficulties with the localization computation, to which we turn now.

3.4 The Localization Computation

Let us briefly summarize where we stand. We have constructed a gauge-fixed action given by a Lagrangian density with five parts: $\mathcal{L}_{v.m.}, \mathcal{L}_{t.c.m.}, \mathcal{L}_{F.I.}, \mathcal{L}_{gh}$, and $\mathcal{L}_W$. This action has a fermionic symmetry $\hat{\mathcal{Q}}$ given as the sum of a generator of the algebra of supersymmetries of the non-gauge-fixed lagrangian and a BRST operator. The first four terms of the Lagrangian density listed above are $\hat{\mathcal{Q}}$ exact and $\hat{\mathcal{Q}}$-closed, so the path integral computing the partition function of our GLSM is independent of the parameters of this part of the Lagrangian, i.e. the partition function depends only on the parameters of the twisted superpotential, which are the complex structure parameters of the quantum Kähler moduli space of the Calabi-Yau to which the GLSM flows in the infrared. We can therefore compute the path integral with any linear combination of the $\hat{\mathcal{Q}}$-exact terms. We choose to compute the path integral with Lagrangian

$$t\,(\mathcal{L}_{v.m.} + \mathcal{L}_{t.c.m.} + i\mathrm{Tr}(\chi D) + \mathcal{L}_{gh}) + \mathcal{L}_W$$

and take the $t \to \infty$ limit. (Here, we've chosen not to include the full Lagrangian $\mathcal{L}_{F.I.}$, but just the actual Fayet-Iliopoulos term; χ^A is a set of couplings for the Fayet-Iliopoulos term). As noted above, when we take the $t \to \infty$ limit, the result is that the path integral reduces to an integral over the minima of the localizing Lagrangian, weighted by a one-loop determinant and the factor $\exp\{-\int d^2x\sqrt{h}\ \mathcal{L}_W\}$. Moreover, as shown in [23], whenever a theory possesses a fermionic symmetry, the integral localizes to the fixed points of that symmetry operator; in the present case, we have both $\mathcal{Q}$ and $\mathcal{Q}_{BRST}$. Finally, we note that we wish to exclude from consideration fixed points with non-zero values for the fermions and the ghost operators of the theory, since we do not want to give vacuum expectation values to Grassmann variables. Keeping this in mind, we examine the bosonic part of the localizing Lagrangian:

$$\begin{aligned}\mathcal{L}_{bos} = \mathrm{Tr}\left((F_{12})^2 + D^2 + i\chi^A D - iD\phi\bar{\phi} + D_\mu\sigma D^\mu\bar{\sigma} + \frac{1}{4}[\sigma,\bar{\sigma}]^2 + \mathcal{G}^2\right) \\ + \bar{F}F + D_\mu\bar{\phi}D^\mu\phi + \bar{\phi}(\sigma_1^2 + \sigma_2^2)\phi.\end{aligned} \tag{3.25}$$

We can let $\tilde{D}^A = D^A + \frac{i}{2}(\chi^A \delta^A - \bar{\phi} t^A \phi)$, with $\delta^A = 1$ when A is an index of a $U(1)$ factor of G and 0 otherwise. Then, the terms involving D can be rewritten as

$$\sum_A \left((\tilde{D}^A)^2 + \frac{1}{4}(\chi^A \delta^A - \bar{\phi} t^A \phi)^2 \right).$$

In this form, the Lagrangian is easily seen to be positive definite with the natural reality conditions imposed on the fields (e.g. $\bar{\phi} = \phi^\dagger$, etc.). Thus, the minima of $\mathcal{L}_{bos}$ are those field configurations which make it 0. This demands, therefore, that:

$$\begin{aligned} F_{12} = \tilde{D} = D_\mu \sigma = D_\mu \bar{\sigma} = [\sigma, \bar{\sigma}] = F = D_\mu \phi \\ = \bar{\phi}(\sigma_1^2 + \sigma_2^2)\phi = \mathcal{G} = \chi^A \delta^A - \bar{\phi} t^A \phi = 0. \end{aligned} \tag{3.26}$$

But, taking into account also that $\mathcal{Q}\lambda = 0$, it follows that $\sigma = \bar{\sigma} = 0$. So the only fields which we do not immediately see to be zero at the minima of $\mathcal{L}_{bos}$ are A_μ and ϕ. We do have, however, that A_μ is a flat connection, though we have not yet specified the gauge-fixing functional $\mathcal{G}$. A usual choice for $\mathcal{G}$ is $\nabla_\mu A^\mu$, which would fix $A_\mu = 0$. There will, however, turn out to be a more convenient choice for $\mathcal{G}$; we will come to it in a moment. Let us first note that $F_{12} = 0$ means that A_μ is gauge equivalent to 0, and therefore that any ϕ satisfying equation 3.26 is gauge-equivalent to a constant configuration (since $D_\mu \phi = 0$). However, even among constant ϕ satisfying $\bar{\phi} t^A \phi = \chi^A \delta^A$ (a D-term equation), there's the residual global gauge symmetry that takes D-term solutions to each other. Thus, up to gauge transformations, the path integral we're considering localizes to the following space $\mathcal{S}$ of zeroes of the action:

$$\mathcal{S} = \{\phi \in \mathfrak{R} \mid \partial_\mu \phi = 0, \quad \bar{\phi} t^A \phi = \chi^A \delta^A\} / G_{global}. \tag{3.27}$$

We want to choose $\mathcal{G}$ so that the satisfaction of $\mathcal{G} = 0$ fixes the gauge completely. This, for example, will not be the case if we choose $\mathcal{G} = \nabla_\mu A^\mu$ since that would only fix $A_\mu = 0$ and leave a residual global gauge freedom in ϕ. Moreover, we can choose $\mathcal{G}$ that will be slightly more convenient when we compute 1-loop determinants; notice that, since the saddle points of the action have a non-zero value for ϕ, the quadratic expansion of the Lagrangian around the saddle points will include terms like $\partial^\mu \bar{\phi} A_\mu \phi_0$, where ϕ_0 is the saddle point and $\bar{\phi}$ and A_μ are the fluctuations around the saddle point configurations. These terms are not particularly convenient for the calculation of one loop determinants, and we can eliminate them by a clever choice of $\mathcal{G}$. To this end, we choose:

$$\mathcal{G}^A = (\nabla_\mu A^\mu)^A - \frac{i}{2}\bar{\phi} t^A \phi_0 + \frac{i}{2}\bar{\phi}_0 t^A \phi. \tag{3.28}$$

This choice of $\mathcal{G}$ requires some explanation, since we must choose $\mathcal{G}$ before we localize, but ϕ_0 makes reference to the saddle points of the action that we examine only once we localize. We have seen, however, that no matter what we choose for $\mathcal{G}$, the path integral localizes to a neighborhood $\mathcal{U}$ in field space of the gauge orbit of $\mathcal{S}$. We can choose $\mathcal{U}$ to be sufficiently small that all fields can be written uniquely as a gauge-fixed saddle point (i.e. element of $\mathcal{S}$) plus a fluctuation (the fluctuation can be both a physical and a gauge fluctuation away from saddle points). This is trivial for all fields but ϕ. For ϕ, we let ϕ_0 denote the projection of ϕ onto $\mathcal{S}$. This is what we mean by ϕ_0 in $\mathcal{G}$. It follows that $\mathcal{G}$ is only defined on $\mathcal{U}$; but that's all right, since the value of $\mathcal{G}$ away from $\mathcal{U}$ is irrelevant. Now, we wish to solve $\mathcal{G} = 0$. We already saw that A_μ is pure gauge and ϕ is gauge equivalent to an element of $\mathcal{S}$. We write $A_\mu = \nabla_\mu \epsilon$ and $\phi = \exp\{i\epsilon^A t_A\}\phi_0$. Then, the condition $\mathcal{G} = 0$ is

$$\nabla^2 \epsilon + \frac{i}{2}\bar{\phi}_0 t^A e^{i\epsilon^B t_B}\phi_0 - \frac{i}{2}\bar{\phi}_0 e^{-i\epsilon^B t_B} t^A \phi_0 = 0. \tag{3.29}$$

Clearly, $\epsilon = 0$ is a solution of this equation; this corresponds to $A_\mu = 0$ and $\phi = \phi_0$. There may, however, be solutions to equation 3.29 which are non-trivial. If the set of solutions is of measure zero with respect to $\mathcal{S}$, then we are justified in ignoring such solutions; this needs to be checked, however. Suppose, to this end, that $\mathcal{G} = 0$ in some finite region around 0 in ϵ-space. Then, the infinitesimal version at $\epsilon = 0$ of equation 3.29 would be satisfied:

$$\nabla^2 \epsilon - \left(\frac{1}{2}\bar{\phi}_0 t^A t^B \phi_0 + \frac{1}{2}\bar{\phi}_0 t^B t^A \phi_0\right)\epsilon_B. \tag{3.30}$$

Now, ∇^2 has a discrete spectrum of eigenvalues on S^2 and the matrix multiplying ϵ_B in parentheses on the RHS of the above equation can easily be checked to be Hermitean. Thus, we can decompose ϵ along an eigenbasis for $\nabla^2 - \frac{1}{2}\left(\bar{\phi}_0 t^A t^B \phi_0 + \bar{\phi}_0 t^B t^A \phi_0\right)$. Since the spectrum of ∇^2 is discrete, the values of ϕ_0 for which we can find a solution to the equation above are at most a discrete subset of $\mathcal{S}$. The case away from $\epsilon = 0$ is nearly identical, except that the infinitesimal expansion around some nonzero ϵ_0 will produce terms like $\bar{\phi}_0 e^{-i\epsilon_0 \cdot t} t^A t^B \phi_0 \epsilon_B$; however, the corresponding matrix will still be Hermitean and have eigenvalues that match ∇^2 only on a discrete subset of $\mathcal{S}$. It follows that, except for a locus of non-zero solutions to $\mathcal{G}^A = 0$ that has measure zero compared to $\mathcal{S}$, the path integral localizes to $\mathcal{S}$. Thus, we can compute

$$Z = \int_{\mathcal{S}} \mathrm{vol}_{\mathcal{S}} e^{-S_W} Z_{1-loop}, \tag{3.31}$$

where Z_{1-loop} is the functional determinant arising from the term quadratic in the fields when the action is expanded around its stationary points and S_W is the superpotential action evaluated at the stationary point. In an Abelian theory with gauge group $U(1)^{N_c}$ and with matter in a N_f-dimensional representation of the gauge group, we can form the $N_f \times N_c$ matrix M whose components are

$$M_I^A = q_I^A (\phi_0)_I.$$

M is a matrix that depends on the choice of $\phi_0 \in \mathcal{S}$. It turns out that $Z_{1-loop} = \det(M^\dagger M)$; we refer the reader to [10] for the details of this computation. We note, however, that our particular choice for $\mathcal{G}$ removes the terms from the quadratic action that couple ϕ and A_μ, so that the quadratic action only couples fields with their Hermitean conjugates. The final answer for the partition function is

$$Z = \int \frac{d^{N_f}\phi \wedge d^{N_f}\bar{\phi}}{(2\pi)^{N_c}} \det(M^\dagger M) \prod_A \delta(\bar{\phi} t^A \phi - \chi^A) e^{S_W(Y,\bar{Y})}, \tag{3.32}$$

where δ is the Dirac delta function which restricts the integral to be only over $\mathcal{S} \times G_{global}$. As mentioned above, the partition function can be used to compute an interesting quantity on the superconformal manifold; we turn now to the elaboration and proof of this statement.

4 Kähler Potential of the Two-Dimensional $\mathcal{N} = (2, 2)$ Superconformal Manifold

It has been shown that, if one restricts to the space $\mathcal{M}$ of two-dimensional, $\mathcal{N} = (2, 2)$ supersymmetric conformal field theories, a manifold structure also arises, and that the metric on the superconformal manifold is Kähler. The manifold is locally a product $\mathcal{S}_c \times \mathcal{S}_{tc}$, and the exactly marginal operators are the top components of chiral and twisted chiral primary operators of $(R, \mathcal{A})$-charges $(2, 0)$ and $(0, 2)$ respectively. In particular, this means that the conformal manifold is complex: heuristically, this corresponds to the pairing of chiral and anti-chiral exactly marginal deformations. The Kähler requirement is that, if one computes the two-form ω defined as follows

$$\omega(X, Y) = g(X, JY),$$

with g the Zamolodchikov metric and J the almost-complex structure on $\mathcal{M}$, then ω is a closed $(1, 1)$ form on $\mathcal{M}$. Moreover, the metric factorizes into a sum $g_c \oplus g_{tc}$, where g_c and g_{tc} are the metric in the chiral and twisted chiral directions.

Two-dimensional CFTs can be canonically placed on a round two-sphere S^2 via stereographic projection in the following way. The pullback of the flat metric on $\mathbb{R}^2$ under the stereographic projection $\pi : S^2 \to \mathbb{R}^2 \cup \{\infty\} = \mathbb{P}^1$ is, up to a conformal factor, the round two-sphere metric:

$$g_*(x) = \frac{1}{\Omega^2(x)} g_{S^2}(x), \tag{4.1}$$

where

$$\Omega(x) = 1 + \frac{|x|^2}{4r^2}. \tag{4.2}$$

In a theory with a Lagrangian description, we can put the theory on the round two-sphere by promoting the flat-space Lagrangian to a diff-invariant one with round two-sphere metric providing the diff-invariant volume element. Classically, the theory retains its conformal symmetry as a Weyl symmetry: under a transformation $g \mapsto \frac{1}{\Omega^2} g$, all the fields transform with Weyl weights equal to their scaling dimensions in the flat-space theory. Under a Weyl symmetry, a correlation function transforms like

$$\langle \phi_1 \cdots \phi_n \rangle \mapsto \Omega^{\frac{1}{2} \sum_{i=1}^{n} \Delta_i} \langle \phi_1 \cdots \phi_n \rangle$$

It is not in general possible to put any CFT on a sphere in this way and preserve all the flat space symmetries. We are interested in the two-dimensional, $\mathcal{N} = (2,2)$ case, and it turns out that it is only possible to preserve one of the two massive $SU(2 \mid 1)$ subalgebras discussed in the previous section and the appendix. Moreover, using the same technique of supersymmetrizing a theory by constructing the appropriate supergravity theory and taking the $M_p \to \infty$ limit, it can be shown (see [2]) that the only ambiguities in the partition function arise from finite supergravity counterterms and they produce the following equivalence:

$$Z \sim Z e^{-F(\lambda) - \bar{F}(\bar{\lambda})},$$

where F is a holomorphic function on the conformal manifold. This is precisely a Kähler ambiguity, and we saw above that the conformal manifold is Kähler in the case at hand, so we might suspect something like

$$Z = e^{-\mathcal{K}},$$

where $\mathcal{K}$ is the Kähler potential of the manifold. Thus, whenever we compute the partition function of a 2D SCFT coupled to S^2 as a function of the λ, as long as we preserve one of the two $SU(2 \mid 1)$'s of the theory, the answer won't depend on the renormalization scheme except potentially through a Kähler ambiguity. In this section, we present a revised version of a proof given in [2] that if we compute the partition function $Z^B_{S^2}$ of a $\mathcal{N} = (2,2)$ CFT on the two sphere while preserving $SU(2 \mid 1)_B$-invariance in a theory formulated with chiral matter (our localization computation above was for the mirror case: $SU(2 \mid 1)_A$ and twisted chiral matter; this gives the same answer), then

$$Z^B_{S^2} = e^{-\mathcal{K}_C}, \tag{4.3}$$

where $\mathcal{K}_C$ is the Kähler potential for g_c. Properly speaking, the partition function ought to be understood not as a function on the conformal manifold, but as a section of a line bundle over $\mathcal{M}$.

To the end of proving equation 4.3, we define the following renormalization scheme:

$$\Gamma_{\mu j}{}^{i} = \begin{cases} H_{\mu j}{}^{i}\,\frac{2^{3-\Delta}r^{2-\Delta}}{2-\Delta} & \Delta_i \neq \Delta_j \\ 0 & \Delta_i = \Delta_j \end{cases} \tag{4.4}$$

where $\Delta = 2 + \Delta_j - \Delta_i$ and r is the radius of S^2. We can see from equation 4.4 that Γ_μ would be singular at $\Delta_i = \Delta_j$; we resolve this problem, however, by demanding that Γ_μ not mix operators of equal dimensions. Let us now say a word about the origin of the weird term involving powers of r and 2 in equation 4.4. We might have naively tried to let

$$\Gamma_\mu \Phi_i = \frac{1}{\pi}\int d^2x\sqrt{h}\ \mathcal{O}_\mu(x)\Phi^i(0) = \frac{1}{\pi}\sum_j H_{\mu i}{}^{j}\Phi_j \int d^2x\sqrt{h}\ \frac{\left(1+\frac{|x|^2}{4r^2}\right)^{\frac{\Delta_\mu+\Delta_i-\Delta_j}{2}-2}}{|x|^{\Delta_\mu+\Delta_i-\Delta_j}},$$

where we have included a normalization factor of $1/\pi$ for simplicity in the final result. The integral on the RHS of the above equation diverges in the ultraviolet for certain values of Δ (where Δ is as defined above), however. We can, though, *define* that integral by analytically continuing in Δ; when $Re(\Delta) < 2$, the integral converges and is equal to precisely the factor multiplying $H_{\mu j}{}^{i}$ in $\Gamma_{\mu j}{}^{i}$ as defined in equation 4.4. That factor makes perfect sense away from $\Delta = 2$, so we have no difficulties in analytically continuing to $\Delta > 2$.

In the case at hand, we have to be sure to address the nuance that we want Γ_μ to preserve $SU(2 \mid 1)_B$; to do this, we let Γ_μ be as above when μ runs over the indices in the chiral direction and zero when μ runs over the indices in the twisted chiral direction. We need $\Gamma_\mu = 0$ for μ in the twisted chiral directions because our procedure is an analytic continuation in the Weyl weight of $\mathcal{O}_\mu$, but we need $\mathcal{O}_\mu$ to have a particular $\mathcal{A}$-charge and primariness dictates an equality between $\mathcal{A}$-charge and Weyl weight. Analytic continuation would therefore break the $SU(2 \mid 1)_B$ symmetry. It can be verified that Γ_μ preserves $SU(2 \mid 1)_B$ symmetry for the μ in the chiral directions. Moreover we can compute the second-order deformation due to Γ_μ according to equation 2.13; since we are computing the Kähler potential we are interested in the identity component of the second-order deformation of the identity operator. First, note that $\Gamma_\mu \mathbb{1} = 4\pi r^2 \mathcal{O}_\mu$. Thus, the term $\partial_\mu \Gamma_\nu$ gives no contribution to the second-order deformation of the identity. We therefore have only the $\Gamma_\mu \Gamma_\nu$ term to consider. The first column of Γ_ν only has a non-zero entry in the row corresponding to $\mathcal{O}_\nu$, and that entry is $4r^2$. Thus, the identity-identity entry of $\Gamma_\mu \Gamma_\nu$ is $4r^2$ times the identity component of

$$\frac{1}{\pi} \int d^2x \sqrt{h}\, \mathcal{O}_\mu(x) \mathcal{O}_\nu(0).$$

But this is precisely $\frac{-1}{4r^2} g_{\mu\nu}$. It follows that the second-order deformation of the partition function is $-g_{\mu\nu}$; note, however, that μ, ν runs only over the indices of the chiral deformations. We can now complete the proof of 4.3:

$$\partial_\mu \partial_\nu \log Z^B_{S^2} = -g_{\mu\nu} = -\partial_\mu \partial_\nu \mathcal{K}_C. \tag{4.5}$$

Equation 4.3 follows from exponentiating both sides and using a supergravity counterterm to set $Z^B_{S^2}$ equal to $e^{-\mathcal{K}_C}$. As mentioned above, we can now use the localization computation for the GLSM above to compute the Kähler potential for the SCFTs which arise as low-energy limits of flat-space gauge theories.

5 Conclusion

We have shown that the structure of the conformal manifold combined with the power of localization turn out to be a very potent combination. Not only can we study certain classes of CFTs wholesale, but we also have means to compute certain quantities of interest to the conformal manifold exactly. There remain, however, many interesting questions in this formalism, the biggest of which is the study of singular points on the conformal manifold.

A deeper understanding of these singularities would reveal more insight about the nature of the conformal manifold. Moreover, the computations here along with those in [9] provide a test for mirror symmetry; these results can be used to find new mirror manifolds.

The conformal manifold formalism provides a new way of structuring and organizing the information that we have on CFTs. The results we have so far are promising, and the hope is that by "zooming out" in this way, we will be able to gain insight about CFTs and quantum field theory in general.

A Spinor Conventions

A.1 Dirac Spinors on S^2

In this section, we review our spinor conventions for spinors on S^2. We will use two-dimensional Dirac spinors, which furnish a reducible representation of the Euclidean version of the 2-d Lorentz group, $U(1)$. We will use μ, ν for two-dimensional space indices, m, n, p for three dimensional indices (when we wish to include the chirality matrix), and α, β for spinor indices. In a frame basis for the tangent space of S^2, we let

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

The γ_μ satisfy $\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu}$, $\mu, \nu = 1, 2$ and γ_3 is the chirality matrix. We also introduce the matrix $C = -i\gamma_2$, which satisfies

$$C\gamma_m = -\gamma_m^T C, \tag{A.2}$$

where $m = 1, 2, 3$, and $P_\pm = \frac{1}{2}\gamma_3(1 \mp \gamma_3)$. $P_\pm$ projects onto subspaces of negative and positive chirality respectively. Anti-commuting spinors can be combined in the following rotationally invariant way:

$$\epsilon\lambda := \epsilon^T C\lambda = \epsilon^\alpha C_{\alpha\beta}\lambda^\beta = \lambda\epsilon. \tag{A.3}$$

We make the following definition:

$$\epsilon\gamma_m\lambda = \epsilon(\gamma_m\lambda) = \epsilon^T C\gamma_m\lambda. \tag{A.4}$$

It can be verified that

$$\epsilon\gamma_m\lambda = -\lambda\gamma_m\epsilon \tag{A.5}$$

for anti-commuting spinors as a consequence of equation A.2. Finally, we mention the Fierz identity, which is useful in verifying the invariance of the GLSM under SUSY transformations:

$$(\epsilon\lambda_1)\,\lambda_2 = -\frac{1}{2}\left((\epsilon\lambda_2)\lambda_1 + (\epsilon\gamma_\mu\lambda_2)\gamma^\mu\lambda_1 + (\epsilon\gamma_3\lambda_2)\gamma^3\lambda_1\right) \tag{A.6}$$

A.2 Conformal Killing Spinors on S^2

We follow the conventions of [9] with our Killing spinors. Explicitly, the space of all ϵ satisfying equation 3.7 is given by:

$$\epsilon = C_1 e^{-i\frac{\phi}{2}} \begin{pmatrix} \sin\theta/2 \\ -i\cos\theta/2 \end{pmatrix} + C_2 e^{i\frac{\phi}{2}} \begin{pmatrix} \cos\theta/2 \\ i\sin\theta/2 \end{pmatrix}, \tag{A.7}$$

where θ, ϕ are the usual polar coordinates on S^2 and $C_{1,2}$ are arbitrary anti-commuting complex parameters.

B Two-dimensional $\mathcal{N} = (2,2)$ superconformal algebra

The $\mathcal{N} = (2,2)$ superconformal algebra can easily be constructed as two copies (one left-moving and one right-moving) of the $N = 2$ superconformal algebra. While this choice of basis for the algebra is easy to construct, it is more natural to use another basis to locate the massive subalgebras that can be preserved when GLSMs are placed on an S^2. This basis consists of the bosonic generators $J_m, K_m, R, \mathcal{A}$ and the supersymmetry generators $Q_\alpha, S_\alpha, \bar{Q}_\alpha, \bar{S}_\alpha$. J_m are the generators of isometries of the sphere, K_m are the generators of conformal transformations of the sphere, and R and $\mathcal{A}$ are the generators of the vectorlike and axial R-symmetries respectively. They satisfy the following commutation relations:

$$
\begin{aligned}
&\{S_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R && [J_m, S^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} S_\beta && [R, S_\alpha] = +S_\alpha \\
&\{\bar{S}_\alpha, \bar{Q}_\beta\} = -\gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta} R && [J_m, Q^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} Q_\beta && [R, Q_\alpha] = -Q_\alpha \\
&\{Q_\alpha, \bar{Q}_\beta\} = \gamma^m_{\alpha\beta} K_m - \frac{i}{2} C_{\alpha\beta} \mathcal{A} && [J_m, \bar{Q}^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{Q}_\beta && [R, \bar{Q}_\alpha] = +\bar{Q}_\alpha \\
&\{S_\alpha, \bar{S}_\beta\} = \gamma^m_{\alpha\beta} K_m + \frac{i}{2} C_{\alpha\beta} \mathcal{A} && [J_m, \bar{S}^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{S}_\beta && [R, \bar{S}_\alpha] = -\bar{S}_\alpha \\
&[J_m, J_n] = i\epsilon_{mnp} J^p && [K_m, S^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{Q}_\beta && [\mathcal{A}, S_\alpha] = i\bar{Q}_\alpha \\
&[K_m, K_n] = -i\epsilon_{mnp} J^p && [K_m, Q^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} \bar{S}_\beta && [\mathcal{A}, Q_\alpha] = -i\bar{S}_\alpha \\
&[J_m, K_n] = i\epsilon_{mnp} K^p && [K_m, \bar{Q}^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} S_\beta && [\mathcal{A}, \bar{Q}_\alpha] = -iS_\alpha \\
& && [K_m, \bar{S}^\alpha] = -\frac{1}{2} \gamma_m{}^{\alpha\beta} Q_\beta && [\mathcal{A}, \bar{S}_\alpha] = iQ_\alpha\,.
\end{aligned}
\tag{B.1}
$$

From these commutation relations, it is easy to see that $J_m, R, Q_\alpha, S_\alpha$ form a subalgebra, which we will denote the $SU(2 \mid 1)_A$ subalgebra, following [10]. Moreover, it can be verified that $J_m, \mathcal{A}$ and the following fermionic generators:

$$
\tilde{S} := \frac{S + \bar{S}}{2} + i\frac{Q + \bar{Q}}{2} \qquad \tilde{Q} := -i\frac{S - \bar{S}}{2} + \frac{Q - \bar{Q}}{2} \tag{B.2}
$$

form another subalgebra with commutation relations

$$
\begin{aligned}
&[J_m, J_n] = i\epsilon_{mnp} J_p && [J_m, \tilde{Q}] = -\frac{1}{2}\gamma_m \tilde{Q} && [J_m, \tilde{S}] = -\frac{1}{2}\gamma_m \tilde{S} \\
&\{\tilde{S}_\alpha, \tilde{Q}_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2} C_{\alpha\beta}\mathcal{A} && [\mathcal{A}, \tilde{Q}] = -\tilde{Q} && [\mathcal{A}, \tilde{S}] = \tilde{S}.
\end{aligned}
\tag{B.3}
$$

We will call this the $SU(2 \mid 1)_B$ subalgebra.

Acknowledgments

I'd like to thank David Kraines and the PRUV program for funding this research during Summer '13; Marco Bertolini for helpful discussions at the initial stages of the project; Paul Aspinwall and Thomas Mehen, the members of my committee, for volunteering their time to read this article; and Ronen Plesser for immeasurable guidance throughout all four of

my undergraduate years.

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