THE CATEGORICAL LANGUAGE OF QUANTUM PHYSICS

EUGENE RABINOVICH

1. Introduction

Category theory has proven to be an important organizer of mathematical knowledge. The field arose in the 1940s as an organizing language for algebraic topology, but has since found wide application in fields of mathematics as diverse as algebra, geometry, topology, and logic. More recently, category theory has also been applied to study computer science and physics. The basic idea is that time and time again in mathematics, we study some sort of mathematical objects and maps, or morphisms between them (e.g. vector spaces and linear maps). And even more important, we often do some sort of alchemy that relates one type of object and morphism to another: we want to build a bridge between two different worlds of objects. This type of bridge is called a functor, and indeed, functors were the main object of focus for the inventors of category theory, who were interested in algebraic objects coming from topological objects (with functors bridging the world between the two). Functorial language is now everywhere in mathematics, and in particular, we'll see how ideas from physics can be packaged into the information of a functor.

On the physics side of things, we're interested in quantum phenomena. Usually, quantum mechanics is described by:

(1) A set V of possible quantum states of a particle. These are described by vectors in a complex inner product space.
(2) A set U_t of operators on V, one for each $t \in \mathbb{R}_{\geq 0}$ satisfying $U_0 = I$ and $U_{t+s} = U_t \circ U_s$. U_t represents the evolution of a state from time 0 to time t.

The physics interpretation of these data is the following: a vector in V represents a possible quantum state a particle can be in. The word "quantum" here means that all phenomena are intrinsically probabilistic, and the inner product encodes all these probabilities. We'll fill in the precise details later, but perhaps it's already clear that quantum mechanics relates the geometry of the non-negative real line to the algebra of vector spaces and inner products. The precise nature of this relationship will be encoded in a functor, as we will see later.

In the rest of the talk, we'll fill in the picture sketched here. First, we'll introduce categories and related concepts; then, we'll talk about quantum mechanics; and finally, we'll show how category theory gives a beautiful language to describe quantum theory by. We'll finish off by saying some words about the current state of the art.

2. Category Theory

A **category** $\mathcal{C}$ consists of the following data:

(1) A collection of objects, denoted $\mathrm{Ob}\,\mathcal{C}$.
(2) For each pair of objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$, a set $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ of morphisms from X to Y. Sometimes we will also write $\mathcal{C}(X, Y)$ for $\mathrm{Mor}_{\mathcal{C}}(X, Y)$.

Date: 29 March 2016.

(3) For each triple $X, Y, Z, \in \text{Ob}\,\mathcal{C}$ and each pair $f \in \text{Mor}(X, Y)$ and $g \in \text{Mor}(Y, Z)$, a composite $g \circ f \in \text{Mor}(X, Z)$, i.e. morphisms are required to compose.
(4) For each $X \in \text{Ob}\,\mathcal{C}$, a morphism $id_X \in \text{Mor}(X, X)$. This is called the identity morphism.

These morphisms are required to satisfy the following:

(1) Whenever $f \in \text{Mor}(X, Y)$ and $g \in \text{Mor}(Y, X)$, $id_X \circ g = g$ and $f \circ id_X = f$.
(2) Whenever the composites make sense,
$$(h \circ g) \circ f = h \circ (g \circ f).$$
This is the associative law for composition.

We will use arrow notation for morphisms, e.g. $f : X \to Y$ means that f is an element of $\text{Mor}(X, Y)$.

The function-like notation for morphisms suggests that objects should be sets with some additional structure and morphisms should be structure-preserving maps. Indeed, we have the following examples of categories:

Example 2.1. The category **Set** whose objects are sets and morphisms are functions between sets. Composition is composition of functions, and the identity morphism is the identity function on X for all sets X.

Example 2.2. The category **Vect** whose objects are vector spaces and morphisms are linear transformations.

Once you have (or if you've had) a little more math, certain other categories of this sort become available to you:

Example 2.3. The category **Grp** whose objects are groups and morphisms are group homomorphisms.

Example 2.4. The category **Rng** whose objects are rings and whose morphisms are ring homomorphisms.

There's a category like for every type of algebraic object one encounters. There are also similar categories coming from topology and geometry. For example:

Example 2.5. The category **Top** whose objects are topological spaces and morphisms are continuous functions.

From these examples alone, one can see that the category concept accommodates many cases arising time and time again in slightly different guises in a lot of fields of mathematics. The definition is flexible enough, however, to allow for even broader applications. Consider the following examples:

Example 2.6. If X is a set, the category $\mathbf{Pow}_X$ whose objects are subsets of X and morphisms are inclusions. That is to say: if U, V are subsets of X, then $\text{Mor}(U, V)$ is a set with one element when $U \subset V$ and otherwise is the empty set. Composition of morphisms is given by composition of inclusions.

Example 2.7. The category whose objects are non-negative numbers and whose morphisms are less-than-or-equal to relations, i.e. if $a, b \in \mathbb{N}$, then $\text{Mor}(a, b)$ is a set with one element when $a \leq b$ and is empty otherwise.

Example 2.8. If Γ is a directed graph, then we can construct a category $\mathcal{C}_\Gamma$ whose objects are the vertices of Γ and whose morphisms are possible paths between pairs of points (the identity morphism at a vertex is the path that just stays at that vertex). Composition of morphisms is simply concatenation of paths end-to-end. (See figure 1).

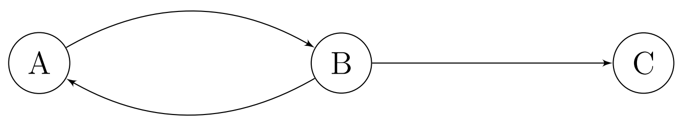

FIGURE 1. A graph Γ gives a category $\mathcal{C}_\Gamma$. $\mathcal{C}_\Gamma$ has three objects: A, B, and C. $\mathcal{C}_\Gamma(A, B)$ is just the single arrow from A to B. $\mathcal{C}_\Gamma(A, A)$ is the set consisting of the constant path at A and of the loop that goes to B and back. $\mathcal{C}_\Gamma(C, A)$ is empty, since there are no paths from C to A.

Example 2.9. The category $\mathbf{Cob}_2$ whose objects are disjoint unions of circles and whose morphisms are surfaces having those circles as a boundary. (See figures 2a and 2b.) Composing morphisms corresponds to gluing the surfaces along the matching boundary component. For example, a cylinder can be seen as a morphism between one circle and another, and a pair of pants is a morphism between two circles and one. If $\mathbf{2}$ is the object of $\mathbf{Cob}_2$ consisting of 2 disjoint circles and similarly for $\mathbf{1}$, then a pair of pants can be seen as an element of $\mathbf{Cob}_2(\mathbf{2}, \mathbf{1})$ or of $\mathbf{Cob}_2(\mathbf{1}, \mathbf{2})$, depending on some subtleties which we brush under the rug here. Let's consider a pair of pants as a morphism $P \in \mathbf{Cob}_2(\mathbf{2}, \mathbf{1})$ and a cylinder as a morphism $C \in \mathbf{Cob}_2(\mathbf{1}, \mathbf{1})$. Then, we can find the composite $C \circ P$, which is a pair of pants with a longer "torso". See figure 2c for another example of a composition of morphisms.

(A) A pair of pants can be considered as an element of $\mathbf{Cob}_2(\mathbf{2}, \mathbf{1})$.

(B) An element of $\mathbf{Cob}_2(\mathbf{2}, \mathbf{2})$.

(C) An example of composition of morphisms in $\mathbf{Cob}_2$.

FIGURE 2. Various morphisms and compositions in $\mathbf{Cob}_2$.

Example 2.10. The category $\mathbf{Cob}'_1$ whose object is a single point and whose morphisms are line segments of arbitrary length connecting two copies of the point (see figure 3). Composition of the morphism of length s and the morphism of length t is the morphism of length $s + t$, i.e. composition is just laying the line segments next to each other.

These last two examples will be crucial for our discussion; $\mathbf{Cob}_2$ models string theory (e.g. a pair of pants is 2 strings colliding and turning into one) and $\mathbf{Cob}'_1$ models particles

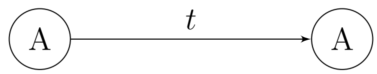

FIGURE 3. A morphism of "length" t in $\mathbf{Cob}_1'$.

in quantum mechanics propagating for different lengths of time (hence the different possible lengths for the line segment.)

We need to introduce one final important idea into our very brief introduction to category theory: the notion of a functor. If $\mathcal{C}$ and $\mathcal{D}$ are two categories, then a **functor** F from $\mathcal{C}$ to $\mathcal{D}$ consists of assignments of:

(1) an object $F(C) \in \text{Ob}(\mathcal{D})$ for every object $C \in \text{Ob}(\mathcal{C})$, and
(2) a morphism $F(f) \in \mathcal{D}(F(C_1), F(C_2))$ for every morphism $f : C_1 \to C_2$ between every pair of objects in $\mathcal{C}$.

These assignments are required to satisfy:

(1) $F(id_C) = id_{F(C)}$ for all $C \in \text{Ob}\,\mathcal{C}$
(2) $F(g \circ f) = F(g) \circ F(f)$ whenever g and f are composable.

Thus, a functor is the categorical version of a function: when we were just dealing with sets, a function was specified by telling where the elements of the domain went. For categories, this isn't enough, since we also have morphisms. A functor provides the necessary additional data, and the requirements above guarantee that a functor preserves the structure of morphism-composition. Another way of putting this is that functors are morphisms between categories. Indeed, there exists a category called **Cat** whose objects are categories and whose morphisms are functors! Though we won't go into this here, **Cat** is best thought of as a 2-category, since we can also define such a thing as a **natural transformation**, which is effectively a morphism of functors, so that **Cat** has objects and morphisms, but also morphisms between morphisms!

We'll finish off with a few easy examples of functor; when we connect physics and category theory, we'll have several more. The first example makes use of the observation that any vector space is in particular a set, and any linear transformation between vector spaces is in particular a function between sets. So we can define a functor **Forget** : **Vect** $\to$ **Set** given by "forgetting" the vector space structure, i.e. sending each vector space to its underlying set and each linear map its underlying map of sets.

For the second example of a functor, let Γ be the directed graph depicted in figure 4. As above, we can form the category C_Γ, which has two objects and one morphism between the two objects. There are also the identity morphisms at A and B. If $\mathcal{C}$ is any category,

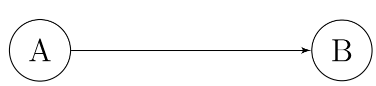

FIGURE 4. The directed graph Γ mentioned above.

then a functor from $\mathcal{C}_\Gamma$ to $\mathcal{C}$ is specified by giving objects $C_A, C_B \in \text{Ob}\,\mathcal{C}$ and morphisms $F(id_A) \in \mathcal{C}(C_A, C_A)$, $F(id_B) \in \mathcal{C}(C_B, C_B)$, and $F(A \to B) \in \mathcal{C}(C_A, C_B)$. Requirement (1) of a functor tells us that $F(id_A) = id_{C_A}$ and $F(id_B) = id_{C_B}$, but $F(A \to B)$ is otherwise unspecified. And requirement (2) is automatically satisfied for all the limited compositions allowed to us since all these compositions involve an identity morphism. Thus, a functor from $\mathcal{C}_\Gamma$ to $\mathcal{C}$ is just the information of a morphism in $\mathcal{C}$. More generally, if Γ is any directed graph, then a functor from $\mathcal{C}_\Gamma$ to $\mathcal{C}$ consists of the information of an object in $\mathcal{C}$ for every vertex of Γ and a morphism between the corresponding objects for every edge in Γ. Since every path in Γ is a composition of edge-long paths, functoriality and associativity guarantee that no more data are needed to specify the functor.

3. Quantum Mechanics

We now turn our attention to the physics part of the talk. In classical mechanics, a particle always has a definite position and momentum, and once you know its position and momentum, you know its position and momentum for all subsequent times. This, however, is no longer true in the quantum mechanical setting. A particle might be in a state of definite position or definite momentum, but it can never be in both a state of definite position and of definite momentum. Moreover, even if a particle is in a definite position at one time, it's rarely the case that it will be in a definite position at subsequent times. Instead, we have to talk about the *probability* of finding a particle in a given position or momentum. The way this works is that the set of possible states of a particle is a complex vector space V; this vector space is endowed with an inner product (which I will denote $\langle , \rangle$) that encodes all the information of probabilities, in the following way. Inside V are states of definite position and momentum (or at least arbitrarily close approximations to such states); if a particle is in state ψ and π is a state with definite momentum p, then the probability that when you measure the particle's momentum you get p is

$$|\langle \psi, \pi \rangle|^2;$$

we say that the inner product measures the **probability amplitude** because its modulus-squared gives the actual probability. Of course, the probability that the particle is in state ψ should be 1, so we need

$$|\langle \psi, \psi \rangle|^2 = 1.$$

i.e. physical states need to have norm 1. Finally, we need to say something about how states evolve with time. This depends on the particulars of the situation, but the time evolution is given by a differential equation. After doing some rigmarole, we find that time evolution is described by a family of operators U_t satisfying $U_{t+s} = U_t \circ U_s$. In other words, if a particle starts in state ψ at a given time, then the particle will be in state $U_t\psi$ after time t has elapsed. The U_t are unitary in the sense that

$$\langle U_t\psi, U_t\phi \rangle = \langle \psi, \phi \rangle,$$

so the U_t's don't change probabilities.

Let's see how this works in a simple example. Let's assume that space is one-dimensional, and in fact a finite box, say that space is just the unit interval $[0, 1]$. Then for V we will take the space of twice differentiable complex-valued functions on $[0, 1]$ that are zero at 0 and 1; this latter requirement is necessary for technical reasons. For an inner product on V we'll take

$$\langle f, g \rangle = \int_0^1 \overline{f(x)} g(x) dx.$$

V does not have states of definite position, but we can approximate such a state by a sequence of states in V. After dealing with this subtlety and some others, we apply the general framework described above to deduce that the probability of finding a particle described by a state $f(x)$ in some region $U \subset [0,1]$ is

$$\int_U |f(x)|^2 dx.$$

Thus, we should interpret $|f(x)|^2$ as the probability density for the particle. (If you're keeping track, the integral that appears in the above expression is not the same one that appears in the definition of the inner product. The former happens because the probability that the particle is in region U is the sum (or really, integral) of the probabilities of the particle being at any one point of U.) Now, if there are no forces on the particle, the differential equation governing the time evolution is just

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi(x,t)}{\partial x^2},$$

where $\hbar$ is Planck's constant, which is basically the constant that's telling us we're doing something quantum. To solve this equation, we can write $\psi(x,t)$ as a Fourier series:

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(2\pi n x),$$

where we've taken into account the assumption that states in V are zero at the boundaries. So the differential equation reduces to

$$\frac{dc_n(t)}{dt} = -\frac{i\hbar(2\pi n)^2}{2}c_n(t).$$

This is solved by

$$c_n(t) = c_n(0)\exp\{-2\pi^2 n\hbar i t\}.$$

In other words, if a state ψ is described by Fourier coefficients c_n, then $U_t\psi$ is described by Fourier coefficients $c_n \exp\{-i\hbar 2\pi^2 n^2 t\}$. It's easily verified that $U_{t+s} = U_t \circ U_s$ and that $U_0 = id_V$.

The important takeaway from all this, which will be useful to us very soon, is that

(1) V is a vector space of possible quantum states of a particle. The inner product on V encodes all the probabilities that you would like to compute.
(2) $\{U_t\}_{t\in\mathbb{R}_{\geq 0}}$ is a set of operators on V, satisfying $U_0 = I$ and $U_{t+s} = U_t \circ U_s$. U_t represents the evolution of a state from time 0 to time t.

This is precisely the setup that will allow us to make the connection to category theory.

4. The Functoriality of Quantum Physics

In this section, we'll see how quantum mechanics is described by a functor F from $\mathbf{Cob}_1'$ to $\mathbf{Vect}$. Indeed, what do we need to do to specify such a functor? Well, first we need an object $F(pt)$ in $\mathbf{Vect}$ corresponding to the single object of $\mathbf{Cob}_1'$. This is precisely a vector space V. Second, we need a morphism $F(t) : F(pt) \to F(pt)$, i.e. an operator on V, for every $t \geq 0$. Let's define $U_t := F(t)$. The definition of composition of morphisms in $\mathbf{Cob}_1'$ was that a morphism of length t composed with a morphism of length s is a morphism of length $t+s$, so requirement (2) of a functor is that

$$U_{t+s} = F(t+s) = F(t)\circ F(s) = U_t \circ U_s,$$

which is precisely the relation we talked about for this family of operators.

Now, you might ask, "Where does the inner product fit into all of this?" The answer is pretty subtle, so we won't get into it here, but suffice it to say that we should think of both $\mathbf{Cob}_1'$ and **Vect** as categories having some more structure (on the **Vect** side, we have the extra structure of inner products) and the functor should preserve this structure. In this formulation, quantum mechanics really can be seen as a special type of functor from $\mathbf{Cob}_1'$ to **Vect**.

The physical intuition here is that the single object in $\mathbf{Cob}_1'$ is a point particle and different morphisms are different lengths of time that the point particle can propagate. If we extend this physical intuition to strings, we see that $\mathbf{Cob}_2$ has as its objects different numbers of strings and as its morphisms different ways these strings can interact with each other and give various numbers of output string states. String theory, then, should be interpreted as a functor from $\mathbf{Cob}_2$ to **Vect**, where the vector spaces describe the quantum states of strings and the operators describe how the possible interaction processes that can happen between strings affect the quantum states.

The really big picture idea here is that, on the one hand, we have the geometry of space-time, given by the categories I've been calling **Cob**, and on the other hand, we have the linear algebra of quantum physics. A functor bridges these two worlds, so is exactly the sort of thing we're looking for to link the two. So, we might want to define a quantum field theory (which includes quantum mechanics as a special case) as a functor between an appropriate **Cob** category and the category of vector spaces. The cool thing about this is that whereas the example we've done is just another way of looking at things that are already well understood, the functorial approach gives us an idea of how to make certain ideas from physics mathematically rigorous and provides a radical reconceptualization of quantum field theory.

5. Frontiers

I've given a cartoon picture of how we should look at quantum mechanics as a functor from a certain geometric category to **Vect**. There are, however, a lot of subtleties involved in figuring out the exact structure of both the geometric category and of **Vect** in all but the simplest cases. Getting these very important details right is a current area of research; in fact, Berkeley's own Peter Teichner has been working this out with his collaborators.

Another interesting and beautiful direction of research has been in generalizing both $\mathbf{Cob}_d$ and **Vect** to d-categories. A d-category is a category that not only has objects and morphisms, but also 2-morphisms which are morphisms between morphisms, 3-morphisms which are morphisms between morphisms between morphisms, and so on up to d-morphisms. Field theories which take this higher-categorical structure into account are called "extended" or "fully local."

A final point that I'd like to make is that this approach is called the *functorial* approach to quantum field theory, for obvious reasons. The other main approach to making sense of quantum field theory is more algebraic in nature and focuses on algebras of observables. One promising new instantiation of this algebraic approach is the formalism of factorization algebras, developed by Kevin Costello and his former student Owen Gwilliam. Peter Teichner and his collaborators have begun to bridge the factorization algebras world and the functorial world, so it's looking like these two approaches really do give something similar.