RESHETIKHIN-TURAEV INVARIANTS OF FRAMED LINKS AND 3-MANIFOLDS

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Today I will tell as much as I can about the invariants Reshetikhin and Turaev constructed based on Ed Witten's seminal physics explorations with quantum Chern-Simons theory. These will be

- An isotopy invariant of framed links.
- An invariant of 3-manifolds.

In the former case, we will construct a special cobordism-like category in which the framed links are the cobordisms from the empty set to the empty set. The invariants of the framed links will arise from a TQFT-like theory we can construct using this category as the domain of an appropriate functor. These invariants of links can then be used to construct an actual 3-dimensional TQFT via surgery, which will give the 3-manifold invariant. Before I give these constructions, I want to make a comment about the physical origins of the theory so that it becomes a little clearer why 1) the invariant is of *framed* links and 2) why representation theory plays such an important role in here. Witten first proposed these invariants as the vacuum expectation values in some quantum gauge theory of the holonomy of particles as they travel around closed loops. In gauge theory, to each particle is attached an irreducible representation of the gauge group G; this is why in all we do below, we will attach representations of some appropriate algebraic object to the connected components of the links. We will actually find that the objects with the right representation theory are called ribon Hopf algebras, so each link component will be assigned a representation of a ribbon Hopf algebra. This explains point 2) above; as for 1), the point is related to the subtle issue of quantizing a classical theory. The quantum field theory that Witten studied was (as such theories usually are) the quantization of a classical theory. There is not usually a unique way to quantize a classical theory; quantization usually requires extra choices. Moreover, not all quantizations of the classical theory would have produced topologically invariant theories. In Witten's work, he found that the quantum theory only assigned a sensible invariant once a framing of the knot was chosen. This is why we'll be studying ribbons and not strings (which you would expect since in this framework we should be thinking of links as trajectories of point particles).

1. The compact braided monoidal category of colored ribbon graphs

Throughout, A will denote an algebra whose category of representations is sufficiently nice in a sense that will become clearer as we proceed and that I'll make precise a bit later.

Definition 1.1. A (k, l)**-ribbon graph** is an oriented surface S consisting of a disjoint union of annuli and squares I^2 embedded in $R^2 \times [0,1]$ such that the bases $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ meet $\mathbb{R}^2 \times [0,1]$ on its boundary at the following collection of points:

$$\{[i - 1/4, i + 1/4] \times 0 \times 0 \mid i = 1, \cdots, k\} \cup \{[i - 1/4, i + 1/4] \times 0 \times 1 \mid i = 1, \cdots, l\}.$$

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The parts of the graph homeomorphic to squares are called the **ribbons** of the ribbon graph and the parts homeomorphic to annuli are just called the **annuli**. The **core** of a ribbon or annulus is $\{1/2\} \times [0,1]$, with or without the identification $0 \sim 1$ made depending as to whether we are considering an annulus or a ribbon respectively.

A ribbon graph is basically a widened version of a tangle (see figure 1 below). We should think of a (k,l)-ribbon graph as a cobordism between an object consisting of k line segments and one consisting of l line segments. Thus, the "in" side is the bottom and the out side is the top. We will, however, need a few fancy-pancifications of the concept of a ribbon graph before we will define the appropriate cobordism category. First, we have

Definition 1.2. A ribbon graph is **directed** if the cores of its ribbons and annuli are provided with directions. For ribbons, a direction gives an initial and final base of the ribbon.

In fact for each line segment $[i-1/4, i+1/4] \times 0 \times 0$ or $[i-1/4, i+1/4] \times 0 \times 1$, we assign a sign ϵ_i or ϵ^i, respectively, where ϵ_i is $+1$ if the corresponding line segment on the bottom boundary of $\mathbb{R}^2 \times [0,1]$ is the final base of a ribbon, and -1 otherwise. Similarly, we let ϵ^i be $+1$ if the corresponding line segment on the top boundary of $\mathbb{R}^2 \times [0,1]$ is the initial base of a ribbon and -1 otherwise. In other words, if we view a ribbon graph as a cobordism from the $z=1$ part to the $z=0$ part, then the sign ϵ^i or ϵ_i is $+1$ if the "in" or "out" designation on the ribbon base coming from the direction of the ribbon matches the "in" or "out" designation coming from the global cobordism assignment of domain and codomain. Figure 1 shows a directed ribbon graph. Next, we have

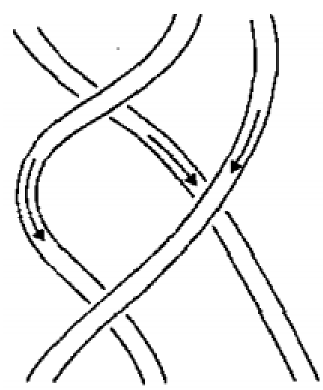


FIGURE 1. A directed ribbon graph

Definition 1.3. A ribbon graph is **homogeneous** if, in a neighborhood of $\mathbb{R}^2 \times \{0,1\}$, the positively oriented normal to the ribbons points in the $-y$ direction, i.e. each ribbon is twisted an even number of times.

We will restrict attention to homogeneous ribbon graphs for simplicity, though this won't be strictly necessary. The final fancy-pancification we will make will be super important. Recall that we were interested in decorating the components of a link with representations of an algebra A. To this end, we introduce:

Definition 1.4. A directed ribbon graph is **colored** when we assign each of its ribbons and annuli a representation of A.

We are now ready to define the cobordism category that we're interested in:

Definition 1.5. The category $HCDR(A)$ has

- as objects, finite sequences $(V_1, \epsilon_1), (V_2, \epsilon_2), \cdots (V_k, \epsilon_k)$ of finite-dimensional representations of A and
- as morphisms between $(V_1, \epsilon_1), (V_2, \epsilon_2), \cdots (V_k, \epsilon_k)$ and $(V^1, \epsilon^1), (V^2, \epsilon^2), \cdots (V^l, \epsilon^l)$ isotopy classes of (k, l) homogeneous, colored, directed ribbon graphs such that the colorings of the ribbons and the induced signs match the labeling of the bases given by the objects, i.e. the ribbon that has a base at $[i - 1/4, i + 1/4] \times 0 \times 1$ should be colored by V^i and the base should be the final or initial base of the ribbon according to whether ϵ^i is -1 or +1, respectively. Isotopies are required to preserve coloring, orientation, direction, and the bases of the ribbons pointwise.

This will be our domain category; the Reshetikhin-Turaev theory will be a functor from $HCDR(A)$ to $RepA$. However, we need to ensure that the category $RepA$ has many of the nice properties of $HCDR(A)$, so we turn to the examination of these properties and the requirements on A so that $Rep(A)$ satisfies these properties.

2. HCDR(A) as a compact braided monoidal category

The first thing that we should notice about $HCDR(A)$ is that it's a monoidal category: the product is given on objects by concatenation of sequences and on morphisms by placing ribbon graphs side by side with no linking or interaction. The corresponding monoidal structure we usually take on **Vect** is tensor product, so we will want to be able to take the tensor product of representations of A.

Next, note that for any pair η, θ of objects in HCDR(A), we have the following morphism $\theta \otimes \eta \to \eta \otimes \theta$: Thus, HCDR(A) has the structure of a braided monoidal category.

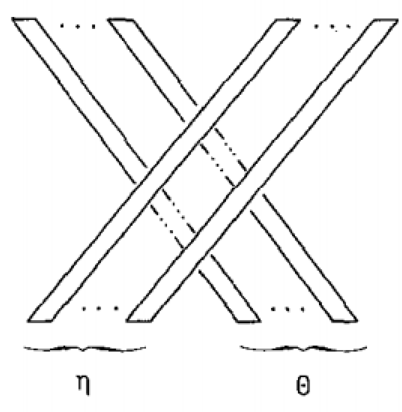
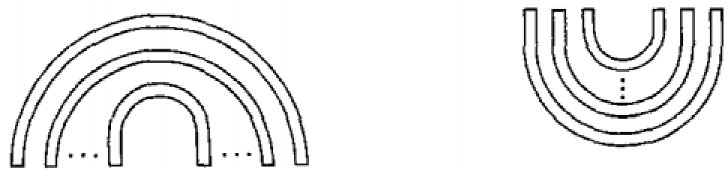


Figure 2. The Braiding in HCDR(A)

Finally, note that $(V_1, \epsilon_1), (V_2, \epsilon_2), \cdots (V_k, \epsilon_k)$ has a dual given by $(V_1, -\epsilon_1), (V_2, -\epsilon_2), \cdots (V_k, -\epsilon_k)$, with unit and counit given by the following diagrams:

These clearly satisfy the zig-zag identity just as they do in our bordism category. The word for a category in which there are duals satisfying zig-zag is compact category. Thus, HCDR(A) is a compact, braided monoidal category. We therefore want $Rep(A)$ to be a



e_η n_η

FIGURE 3. The Compact Categorical Structure Maps in HCDR(A)
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compact braided monoidal category. We proceed to examine the conditions on A that guarantee this to be the case.

3. Ribbon Hopf Algebras

We will proceed by introducing a succession of more involved structures on an algebra that will guarantee that the category $RepA$ has a monoidal product, duals, and a braiding. We will start with the monoidal product. Recall that an algebra is a ($\mathbb{C}$) vector space A together with maps

$$\mu : A \otimes A \to A$$
$$\eta : \mathbb{C} \to A$$

called multiplication and unit, respectively. These maps satisfy associativity and unit relations that can easily be drawn as commutative diagrams. If A is an algebra and ρ_V, ρ_W are representations of A on V and W respectively, there is no canonical way to put the structure of A representation on $V \otimes W$. The issue is that the naive thing to do, to take $\rho_{V\otimes W}(a)(v \otimes w) = \rho_V(a)v \otimes \rho_W(a)w$, is not linear in A and actually gives a representation of $A \otimes A$. So, if we have an algebra map $A \times A \otimes A$ (where $A \otimes A$ is given the natural algebra structure), we will be able to form tensor products in $RepA$. Moreover, to have a monoidal unit, we will need a representation of A on $\mathbb{C}$, which should be given by a map $A \to \mathbb{C}$. Notice that what we need is maps going in the opposite direction as μ and η. We are therefore motivated to make the following definition:

Definition 3.1. An algebra A is a **bialgebra** if in addition to its maps μ and η, it posseses algebra maps

$$\Delta : A \to A \otimes A$$
$$\epsilon : A \to \mathbb{C},$$

such that Δ and ϵ satisfy coassociativity and counit relations.

Theorem 3.1. *If A is a bialgebra, then $RepA$ is a monoidal category with monoidal product given by tensor products of vector spaces and unit given by $\mathbb{C}$. These vector spaces are given the structure of A representations with the following maps:*

$$\rho_{V\otimes W}(a) = \rho_V \otimes \rho_W(\Delta(a))$$
$$\rho_{\mathbb{C}}(a)(z) = \epsilon(a)z$$

Next, we add the structure of duals to the mix. Notice that if A is an algebra and (V, ρ_V) is a representation, then the naive way to put the structure of A-representation on $V^\vee$, namely that $\rho_{V^\vee}(a) = \rho_V(a)^\vee$ doesn't work because taking duals is contravariant. This issue doesn't come up in groups where inversion gives a map $G \to G$ that reverses the order of products, so perhaps we should be looking for a similar picture in the world of bialgebras. If A is a bialgebra, we can put the structure of algebra on $Hom_{\mathbb{C}}(A, A)$ as follows. Given $f, g \in Hom_{\mathbb{C}}(A, A)$, define $f \star g$ as the following map:

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A.$$

With the unit $\eta \circ \epsilon$, this is an associative algebra.

Definition 3.2. A bialgebra A is a **Hopf algebra** if there exists an inverse S for the identity map on A for the $\star$ multiplication, i.e. if there exists and S satisfying

$$S \star id_H = id_H \star S = \eta \circ \epsilon.$$

Cool fact: an antipode is an anti-algebra map: $S(ab) = S(b)S(a)$. Thus, if A is a Hopf algebra and (V, ρ_V) is a representation of A, then we can define the structure of an A-representation on $V^\vee$ by

$$\rho_{V^\vee}(a) = \rho_V(S(a))^\vee.$$

Aside: A group can be considered a Hopf algebra over $\mathbb{F}_1$

Theorem 3.2. *For A a Hopf algebra, the category of representations of A is a compact monoidal category with the structures defined above, along with the usual evaluation and coevaluation maps for V and its dual.*

Finally, we add the braiding. If A is a Hopf algebra and R is an invertible element of $A \otimes A$ then, letting $P : A \otimes A \to A \otimes A$ be the flip homomorphism, we make the following definition:

Definition 3.3. The pair (A, R) is a **quasitriangular Hopf algebra** precisely when

$$\Delta'(a) = R\Delta(a)R^{-1}$$
$$(\Delta \otimes id_A)(R) = R_{12}R_{23}$$
$$(id_A \otimes \Delta)(R) = R_{12}R_{12},$$

where $\Delta' = P \circ \Delta$, $R_{12} = R \otimes 1$, $R_{12} = (id \otimes P)(R_12)$, $R_{23} = 1 \otimes R$.

The first equation just says that conjugation by R effects the swap homomorphism on coproducts, while the last two equations guarantee the following theorem:

Theorem 3.3. *For A a quasitriangular Hopf algebra, the category of representations of A is a compact braided monoidal category with the braiding defined by*

$$c_{V,W} = P^{V,W} \circ (\rho_V \otimes \rho_W)(R)$$

So quasitriangular Hopf algebras give us the full compact braided monoidal structure that we should hope for $RepA$. However, we will need one last technical assumption on A before we can construct the Reshetikhin-Turaev invariant associated to framed links. If (A, R) is a QTHA, we write $R = \sum_i \alpha_i \otimes \beta_i$ and we let

$$u = (\mu \circ P \circ (id \otimes S))(R) = \sum_i S(\beta_i) \otimes \alpha_i.$$

Definition 3.4. A QTHA (A,R), along with a central element $v \in A$ is called a **Ribbon Hopf algebra** if $v^2 = uS(u)$ and v satisfies a number of other equations which we won't write out.

Finally, we have found a class of algebras whose representation category will be the codomain of our TQFT-like functor. In fact, we have the following theorem.

Theorem 3.4. *Let (A, R, v) be a ribbon Hopf algebra. Then there exists a unique functor $HCDR(A) \to RepA$ satifsying*

1. *F preserves the monoidal product on both categories.*
2. *F takes the object (V, ϵ) to V if $\epsilon = +1$ and $V^\vee$ if $\epsilon = -1$.*
3. *F takes the caps pictured below to the maps*
$$(x, y) \mapsto x(y) : V^\vee \otimes V \to \mathbb{C}$$
and
$$(y, x) \mapsto x(v^{-1}uy) : V \otimes V^\vee \to \mathbb{C},$$
respectively.
4. *F takes the X graph pictured below to the map*
$$x \otimes y \mapsto \sum_i \beta_i y \otimes \alpha_i x = (P \circ (\rho_V \otimes \rho_W(R)))(x \otimes y).$$

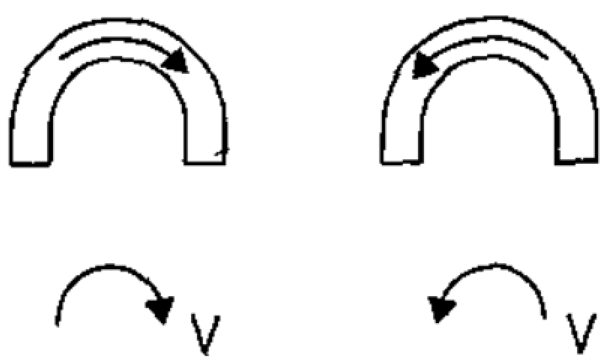

FIGURE 4. Caps

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If L is a framed link in $\mathbb{R}^3$, the framing is determined by a normal vector to L. Using this normal vector to widen L, we get a $(0, 0)$-ribbon graph. If we choose a labeling of L, F gives us a well-defined isotopy invariant of links. This is the famed Reshetikhin-Turaev invariant of framed links.

4. RT invariant of 3-Manifolds

I will make some very brief comments about how to use this theory to give the actual Reshetikhin-Turaev TQFT, or at least how to get the 3-manifold invariant. It is a theorem that any closed oriented 3-manifold can be obtained from S^3 by cutting out a tubular neighborhood of a link and gluing back in a solid torus with new gluing data. The new gluing data are specified by a framing of L. Moreover, L can be isotoped into $\mathbb{R}^2 \times [0, 1]$, in which

FIGURE 5. Crossings

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case the theory above gives us a well-defined number associated to the link L with the given framing. Roughly, this idea is used to construct the 3-manifold invariant.