

Name: _____

Section: _____

1. Find the general solution to the differential equation $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\chi_A(\lambda) = \lambda^2 - 4\lambda + 5$$

$$\lambda = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$$

$$\lambda = 2 + i: \begin{bmatrix} 4 - i & 1 \\ -1 & 4 - i \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 + (1-i)R_2} \begin{bmatrix} 0 & 0 \\ -1 & 4 - i \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -i \\ 1 \end{bmatrix} \text{ is an e-vec}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix} \rightarrow e^{2t} \left(\cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$+ e^{2t} \left(\sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

are solns. Gen & Soln:

$$c_1 e^{2t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

2. Find the solution of the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = (1, -1)^T$, where

$$A = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix}.$$

$$\chi_A(\lambda) = (\lambda + 1)^2 \rightarrow \text{Need to find Null}(A + I)^2$$

$$A + I = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \quad (A + I)^2 = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{So, } e^{At} = e^{(A+I)t - It} = e^{-t} e^{(A+I)t}$$

$$= e^{-t} \left(I + t \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} + \frac{t^2 (A+I)^2}{2!} + \dots \right)$$

$$= e^{-t} \begin{bmatrix} 1 & 0 \\ -2t & 1 \end{bmatrix}$$

$$\mathbf{y}(t) = e^{At} \mathbf{y}(0) = e^{-t} \begin{bmatrix} 1 & 0 \\ -2t & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} 1 \\ -2t - 1 \end{bmatrix}$$