

Name: _____
 Section: Solutions

1. Let

$$A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}.$$

Using the fact that $\mathbf{v}_1 = (-2, 2, 1)^T$ and $\mathbf{v}_2 = (1, 1, 0)^T$ are eigenvectors for A , find an orthogonal basis for \mathbb{R}^3 consisting of eigenvectors for A . (Hint: You don't need to solve the characteristic equation or find eigenspaces in this problem.)

As mentioned in section, it suffices to find

a vector $\underline{\mathbf{v}}_3$ that is orthogonal to $\underline{\mathbf{v}}_1$ and $\underline{\mathbf{v}}_2$ and that vector will automatically be an eigenvector for A : Thus, if

$$\underline{\mathbf{v}}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \text{ we have } \begin{aligned} -2a+2b+c &= 0 \\ a+b &= 0 \end{aligned} \rightarrow \begin{bmatrix} -2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \end{bmatrix} \rightarrow \underline{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \text{ is such a vector and } A\underline{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \underline{\mathbf{v}}_3, \text{ so } \underline{\mathbf{v}}_3 \text{ is an e-vec.}$$

2. Let $\mathcal{A} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be two bases for a vector space V , and suppose

$$\mathbf{u}_1 = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3, \quad \mathbf{u}_2 = 3\mathbf{v}_1 - \mathbf{v}_2, \quad \mathbf{u}_3 = -\mathbf{v}_2 + \mathbf{v}_3.$$

Let $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$. Find $[\mathbf{v}]_{\mathcal{A}}$.

$$\underline{\mathbf{v}} = 2\underline{\mathbf{v}}_1 - 3\underline{\mathbf{v}}_2 + \underline{\mathbf{v}}_3 + 3\underline{\mathbf{v}}_1 - \underline{\mathbf{v}}_2 - \underline{\mathbf{v}}_2 + \underline{\mathbf{v}}_3$$

$$= 5\underline{\mathbf{v}}_1 - 5\underline{\mathbf{v}}_2 + 2\underline{\mathbf{v}}_3$$

$$\Rightarrow [\underline{\mathbf{v}}]_{\mathcal{A}} = \boxed{\begin{bmatrix} 5 \\ -5 \\ 2 \end{bmatrix}}$$

Thus, the basis is

$$\boxed{\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}}$$