Worksheet 8: February 14

1 Divisors

1. List all divisors of each of the following numbers.

   (a) 6: 1, 2, 3, 6
   (b) 13: 1, 13
   (c) 18: 1, 2, 3, 6, 9, 18
   (d) 25: 1, 5, 25
   (e) 81: 1, 3, 9, 81
   (f) 100: 1, 2, 4, 5, 10, 20, 25, 50, 100

2. From the previous question: For each of the following pairs of integers, find their greatest common divisor. Which pairs are *relatively prime*?

   (a) 13 and 25: 1 (relatively prime)
   (b) 6 and 81: 3
   (c) 25 and 100: 25
   (d) 6 and 25: 1 (relatively prime)
   (e) 18 and 81: 9

2 The Euclidean algorithm and the Chinese remainder theorem

3. (a) Use the Euclidean algorithm to find the GCD of 270 and 192.

$$\begin{aligned} 270 &= 1 \cdot 192 + 78 \\ 192 &= 2 \cdot 78 + 36 \\ 78 &= 2 \cdot 36 + 6 \\ 36 &= 6 \cdot 6 + 0 \end{aligned}$$

   The GCD is 6.

   (b) Let d be the GCD you just found. Work through the steps of the algorithm to find integers a and b such that $270a + 192b = d$.

$$\begin{aligned} 78 &= 270 - 192 \\ 36 &= 192 - 2 \cdot 78 = 192 - 2(270 - 192) = -2 \cdot 270 + 3 \cdot 192 \\ 6 &= 78 - 2 \cdot 36 = (270 - 192) - 2(-2 \cdot 270 + 3 \cdot 192) = 5 \cdot 270 - 7 \cdot 192 \end{aligned}$$

   $a = 5$, $b = -7$

4. (a) Use the Euclidean algorithm to find the GCD of 13 and 98.

$$\begin{aligned} 98 &= 7 \cdot 13 + 7 \\ 13 &= 1 \cdot 7 + 6 \\ 7 &= 1 \cdot 6 + 1 \end{aligned}$$

The GCD is 1.

(b) Let d be the GCD you just found. Work through the steps of the algorithm to find integers a and b such that $13a + 98b = d$.

$$\begin{aligned} &7 = 98 - 7 \cdot 13 \\ &6 = 13 - 1 \cdot 7 = 13 - (98 - 7 \cdot 13) = -98 + 8 \cdot 13 \\ &1 = 7 - 1 \cdot 6 = (98 - 7 \cdot 13) - (-98 + 8 \cdot 13) = 2 \cdot 98 - 15 \cdot 13 \end{aligned}$$

$a = 2$, $b = -15$

5. For each of the following sets of congruences, find *all* integers x such that every congruence holds.

(a) $\begin{cases} x \equiv 2 \mod 5 \\ x \equiv 3 \mod 7 \end{cases}$

Solution: $x \equiv 17 \mod 35$, so $x = 17 + 35k$ for any $k \in \mathbb{Z}$

(b) $\begin{cases} x \equiv 1 \mod 2 \\ x \equiv 2 \mod 3 \\ x \equiv 3 \mod 5 \end{cases}$

Solution: First reduce to $x \equiv 5 \mod 6$ $x \equiv 3 \mod 5$, then combine to get $x \equiv 23 \mod 30$, so $x = 23 + 30k$ for any $k \in \mathbb{Z}$.

(c) $\begin{cases} x \equiv 8 \mod 13 \\ x \equiv 76 \mod 98 \end{cases}$

Solution: We know from problem 4 that $2 \cdot 98 - 15 \cdot 13 = 1$, so $2 \cdot 98 \equiv 1 \mod 13$ and $-15 \cdot 13 \equiv 1 \mod 98$. Let $n = 8 \cdot (2 \cdot 98) + 76 \cdot (-15 \cdot 13)$. Then $x = n + (13 \cdot 98)k$ for any $k \in \mathbb{Z}$.

(d) $\begin{cases} x \equiv 1 \mod 6 \\ x \equiv 2 \mod 10 \end{cases}$

Solution: The first equivalence implies $x \equiv 1 \mod 2$, but the second implies $x \equiv 0 \mod 2$. No such x exists.