Worksheet 6: February 7 (Solutions)

1 Cardinality

1. For each of the following sets A, determine whether A is finite, countable, or uncountable. Provide an (informal) proof for each.

   (a) $A = \{2, 6, 19\}$
   Finite.

   (b) $A = \mathbb{Z}$
   Countable.

   (c) $A = \{\text{prime numbers}\}$
   Countable.

   (d) $A = \mathbb{Q}$
   Countable.

   (e) $A = \{\text{atoms in the universe}\}$
   Finite.

   (f) $A = \mathbb{Z} \times \mathbb{Z}$
   Countable.

   (g) $A = \mathbb{R}$
   Uncountable.

   (h) $A = \{\text{subsets of } \mathbb{Z}\}$
   Uncountable.

2. For each of the following pairs of sets, determine which has larger cardinality. Find an injective map from the smaller map to the larger one. If they have the same cardinality, try to find a bijective map between them.

   (a) $A = \{0, 2, 5\}$, $B = \{1, 3\}$
   $|A| > |B|$. Map: $1 \mapsto 0$, $3 \mapsto 5$

   (b) $A = \{\text{submitted Math 55 Homework 1s}\}$, $B = \{\text{students in Math 55}\}$
   $|B| > |A|$. Map: (homework submission) $\mapsto$ (student who submitted it)

   (c) $A = \{\text{people in the world}\}$, $B = \{\text{human brain cells}\}$
   $|B| > |A|$. Map: (person) $\mapsto$ (one of that person's brain cells)

   (d) $A = \{\text{books}\}$, $B = \{\text{authors}\}$
   $|A| > |B|$. Map: (author) $\mapsto$ (that author's first book)

   (e) $A = \{\text{positive integers}\}$, $B = \{\text{positive even integers}\}$
   $|A| = |B| = \aleph_0$. Map: $x \mapsto 2x$

(f) $A = \{\text{positive integers}\}$, $B = \{\text{prime numbers}\}$
$|A| = |B| = \aleph_0$. Map: $x \mapsto p_x$

(g) $A = \mathcal{P}(\{1, 2, \ldots, n\})$, $B = \{\text{binary strings of length } n\}$
$|A| = |B| = 2^n$. Map: $S \to b_1 b_2 \cdots b_n$, where $b_k = 1$ if $k \in S$ and $b_k = 0$ if $k \notin S$

(h) $A = \mathbb{R}$, $B = \mathbb{Z}$
$|A| > |B|$. Map: $x \mapsto x$

(i) $A = \mathbb{Q}, B = \mathbb{Q} \times \{0, 1, 2\}$
$|A| = |B| = \aleph_0$. Order the rationals first: $q_1, q_2, q_3, \ldots$ Then order B: $(q_1, 0)$, $(q_1, 1)$, $(q_1, 2)$, $(q_2, 0)$, $(q_2, 1)$, $(q_2, 2)$, $(q_3, 0)$, $(q_3, 1)$, $(q_3, 2), \ldots$ Assign the k-th element of this latter sequence to q_k.

(j) $A = \mathbb{Z}$, $B = \mathbb{Z} \times \mathbb{Z}$
$|A| = |B| = \aleph_0$. Diagonal map.

(k) $A = \mathbb{Z} \times \mathbb{Z}$, $B = \mathbb{Q}$
$|A| = |B| = \aleph_0$. Map the rationals to the integers first, then diagonal map.

(l) $A = \mathbb{Z}$, $B = \mathcal{P}(\mathbb{Z})$
$|A| < |B|$. Map: $x \to \{x\}$