Worksheet 4: January 31 (Solutions)

1 Arguments and Proofs, part 2

1. Given any n real numbers $a_1, a_2, \ldots, a_n$, prove that at least one of them is greater than or equal to the average of these numbers.
   Solution: Assume the opposite: for all k, $a_k < \frac{a_1+\cdots+a_n}{n}$. Sum over all k to get $a_1 + \cdots + a_n < \frac{a_1+\cdots+a_n}{n} + \cdots + \frac{a_1+\cdots+a_n}{n} = n\frac{a_1+\cdots+a_n}{n} = a_1 + \cdots + a_n$.

2. Use the previous exercise to show that if the first 10 positive integers are placed around a circle, in any order, there exist three numbers that are adjacent on the circle and sum to at least 17.
   Solution: Let $r_1, r_2, \ldots, r_{10}$ be an ordering of the first 10 positive integers. Then the orderings of the three consecutive integers are $a_1 = r_1 + r_2 + r_3$, $a_2 = r_2 + r_3 + r_4$, $a_3 = r_3 + r_4 + r_5$, ..., $a_8 = r_8 + r_9 + r_{10}$, $a_9 = r_9 + r_{10} + r_1$, and $a_10 = r_{10} + r_1 + r_2$. Each of the first ten positive integers is represented exactly three times in the a_k's, so the sum of the a_k's is $3(1 + 2 + \cdots + 10) = 3 \cdot 55 = 165$. The average of the a_k's is 16.5, so there must be some a_k that is at least 16.5; the smallest integer that is at least 16.5 is 17.

3. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$. [Hint: Assume that $r = a/b$ is a root, where a and b are integers and a/b is in lowest terms. Obtain an equation involving integers by multiplying by b^3. Then look at whether a and b are each odd or even.]
   Solution: From the assumption, we have $\frac{a^3}{b^3} + \frac{a}{b} + 1 = 0$, and so $a^3 + ab^2 + b^3 = 0$. The left-hand side is only even when both a and b are even, and so $\frac{a}{b}$ is not in lowest terms.

4. Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|\cdot|$ represents absolute value).
   Solution: Proof by cases. Either $|x + y| = x + y$, or $|x + y| = -x - y$, and either case is $\leq |x| + |y|$.

5. Let $A = 65^{1000} - 8^{2001} + 3^{177}$, let $B = 79^{1212} - 9^{2399} + 2^{2001}$, and let $C = 25^{449} - 5^{8192} + 7^{1777}$. Show that at least one of AB, AC, and BC is nonnegative.
   Solution: Among *any* three numbers, at least one pair must multiply to a nonnegative number.

6. (Challenge) Prove that between every two rational numbers there is an irrational number.
   Solution: For any two distinct rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, the irrational number $\frac{a}{b} + \frac{1}{\sqrt{2}}(\frac{c}{d} - \frac{a}{b})$ lies between them.

2 Sets, part 1

7. For each of the following sets A, give an example of a finite subset $B \subsetneq A$ and an infinite set $C : A \subsetneq C$.

   - $A = \{2, 3, 5, 7, 11\}$, $B = \{2, 7\}$, $C = \{\text{prime numbers}\}$
   - $A = \mathbb{Z}$, $B = \{2, 3, 5, 7, 11\}$, $C = \mathbb{Q}$
   - $A = \{1\}$, $B = \emptyset$, $C = \mathbb{Z}$
   - $A = \{\text{people in this room}\}$, $B = \{\text{Jacob}\}$, $C = \{\text{people in this room}\} \cup \mathbb{R}$
   - $A = \mathbb{R}$, $B = \{2, 3, 5, 7, 11\}$, $C = \mathbb{C}$
   - $A = \{x \in \mathbb{R} : |x| < 1\}$, $B = \{0\}$, $C = \mathbb{R}$
   - $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y\}$, $B = \{(1, 3), (2, 7)\}$, $C = \mathbb{R} \times \mathbb{R}$

8. Determine the truth value of the following statements. Give a proof for each.

   - $\forall A \forall B : A \cap B \subseteq A \subseteq A \cup B$
     True. Trivial.
   - $\forall A \forall B : (A \setminus B) \cup B = A$
     False. For $A = \{1\}$ and $B = \{2\}$, $(A \setminus B) \cup B = \{1, 2\}$.
   - $\forall$ finite nonempty $A \subseteq \mathbb{Z}, \exists x \in A : A \subseteq \{y \in \mathbb{Z} : y \leq x\}$
     True for $x = \max(A)$.
   - $\forall$ infinite $A \subseteq \mathbb{Z}, \exists x \in A : \{y \in \mathbb{Z} : y \leq x\} \subseteq A\}$
     False. $\mathbb{Z}$ itself is a counterexample.
   - $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \exists A \subseteq \mathbb{Z} : x \in A, y \notin A$
     False. No A exists when $x = y$.
   - $\exists A \subseteq \mathbb{Z}, \forall B \subseteq \mathbb{Z} : A \cap B = B$
     True. $A = \mathbb{Z}$
   - $\exists A \subseteq \mathbb{Z}, \forall B \subseteq \mathbb{Z} : A \cap B = A$
     True. $A = \emptyset$