Worksheet 4: January 31

1 Arguments and Proofs, part 2

1. Given any n real numbers $a_1, a_2, \ldots, a_n$, prove that at least one of them is greater than or equal to the average of these numbers.
   Solution:

2. Use the previous exercise to show that if the first 10 positive integers are placed around a circle, in any order, there exist three numbers that are adjacent on the circle and sum to at least 17.
   Solution:

3. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$. [Hint: Assume that $r = a/b$ is a root, where a and b are integers and a/b is in lowest terms. Obtain an equation involving integers by multiplying by b^3. Then look at whether a and b are each odd or even.]
   Solution:

4. Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|\cdot|$ represents absolute value).
   Solution:

5. Let $A = 65^{1000} - 8^{2001} + 3^{177}$, let $B = 79^{1212} - 9^{2399} + 2^{2001}$, and let $C = 25^{449} - 5^{8192} + 7^{1777}$. Show that at least one of AB, AC, and BC is nonnegative.
   Solution:

6. (Challenge) Prove that between every two rational numbers there is an irrational number.
   Solution:

2 Sets, part 1

7. For each of the following sets A, give an example of a finite subset $B \subsetneq A$ and an infinite set $C : A \subsetneq C$.

   - $A = \{2, 3, 5, 7, 11\}$
   - $A = \mathbb{Z}$
   - $A = \{1\}$
   - $A = \{\text{people in this room}\}$
   - $A = \mathbb{R}$
   - $A = \{x \in \mathbb{R} : |x| < 1\}$
   - $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y\}$

8. Determine the truth value of the following statements. Give a proof for each.

   - $\forall A \forall B : A \cap B \subseteq A \subseteq A \cup B$
   - $\forall A \forall B : (A \setminus B) \cup B = A$
   - $\forall$ finite nonempty $A \subseteq \mathbb{Z}, \exists x \in A : A \subseteq \{y \in \mathbb{Z} : y \leq x\}$
   - $\forall$ infinite $A \subseteq \mathbb{Z}, \exists x \in A : \{y \in \mathbb{Z} : y \leq x\} \subseteq A\}$
   - $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \exists A \subseteq \mathbb{Z} : x \in A, y \notin A$
   - $\exists A \subseteq \mathbb{Z}, \forall B \subseteq \mathbb{Z} : A \cap B = B$
   - $\exists A \subseteq \mathbb{Z}, \forall B \subseteq \mathbb{Z} : A \cap B = A$