Worksheet 3: January 29 (Solutions)

1 Arguments and Proofs, part 1

1. You are given these two assumptions:

   (i) "Logic is difficult or not many students like logic."

   (ii) "If mathematics is easy, then logic is not difficult."

   By translating these assumptions into statements involving propositional variables and logical connectives, determine whether each of the following are valid conclusions of these assumptions:

   (a) Mathematics is not easy if many students like logic.

   (b) Not many students like logic if mathematics is not easy.

   (c) Mathematics is easy or logic is difficult.

   (d) Logic is not difficult or mathematics is not easy.

   (e) If not many students like logic, then *either* mathematics is not easy or logic is not difficult.

   Solution: Let p be "logic is difficult", let q be "many students like logic", and let r be "mathematics is easy." Our assumptions are $p \to \neg q$ and $r \to \neg p$.

   (a) $q \to \neg r$ follows from the contrapositives of both assumptions.

   (b) $\neg r \to \neg q$ does not follow.

   (c) $r \vee p$ does not follow.

   (d) $\neg p \vee \neg r$ follows from the second assumption.

   (e) $\neg q \to (\neg r \oplus \neg p)$ does not follow.

2. What is wrong with this argument? Let $S(x, y)$ be "x is shorter than y". Given the premise $\exists s\, S(s, \text{Max})$, it follows that $S(\text{Max}, \text{Max})$. Then by existential generalization it follows that $\exists x\, S(x, x)$, so that someone is shorter than themself.
   Solution: Universal instantiation doesn't work on an existential quantifier.

3. Use a direct proof to show that the product of two rational numbers is rational.
Solution: If $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, then $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some integers a, b, c, d such that $b \neq 0$ and $d \neq 0$. Then $xy = \frac{ac}{bd}$. Because $ac \in \mathbb{Z}$, $bd \in \mathbb{Z}$, and $bd \neq 0$, $xy \in \mathbb{Q}$.

4. Disprove that the product of two irrational numbers is irrational.
Solution: Counterexample: $\sqrt{2} \times \sqrt{2} = 2 \in \mathbb{Q}$.

5. Prove that if m and n are integers and mn is even, then m is even or n is even.
Solution: Proof by contrapositive: if m and n are both odd, then mn is odd. If m and n are both odd, then $m = 2x+1$ and $n = 2y+1$ for some integers x and y. Then $mn = (2x+1)(2y+1) = 2(2xy+x+y)+1$, which is odd.

6. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.
Solution: Proof by contradiction: Assume that an irrational number x exists such that its product with a nonzero rational number $\frac{a}{b}$ is a rational number $\frac{c}{d}$. Then $x = \frac{bd}{ac}$, which is rational.

7. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.
Solution: You can either get three blue socks, two blue and one black, one blue and two black, or three black. In the first two cases, you have a pair of blue socks; in the last two cases, you have a pair of black socks.