Worksheet 2: January 24 (Solutions)

1 Predicates and Quantifiers

1. What are the truth values of these statements?

   (a) $\exists x\, P(x) \to \forall x\, P(x)$

   (b) $\forall x\, P(x) \to \exists x\, P(x)$

   (c) $\exists x\, \neg P(x) \to \neg \forall x\, P(x)$

   (d) $\neg \forall x\, P(x) \to \neg \exists x\, P(x)$

   (e) $\neg \forall x\, P(x) \to \exists x\, \neg P(x)$

   (f) $\neg \exists x\, P(x) \to \forall x\, \neg P(x)$

   Solution: (a) is false, (b) is true, (c) is true, (d) is false, (e) is true, (f) is true.

2. Find a counterexample to the following statements:

   (a) All prime numbers are even.

   (b) The square of a positive integer x is larger than x.

   Solution: (a) 3 is prime but odd. (b) $1^2 \not> 1$.

2 Nested Quantifiers

3. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

   (a) $\forall n \exists m (n^2 < m)$

   (b) $\exists n \forall m (n < m^2)$

   (c) $\forall n \exists m (n + m = 0)$

   (d) $\exists n \forall m (nm = m)$

   (e) $\exists n \exists m (n^2 + m^2 = 5)$

   (f) $\exists n \exists m (n^2 + m^2 = 6)$

   Solution: (f) is false, the other five are true.

3 Rules of Inference

4. Determine whether these are valid arguments.

   (a) If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.

   (b) If $x^2 \neq 0$, where x is a real number, then $x \neq 0$. Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.

   Solution: The first is false: a statement does not imply its inverse. The second is true: it's an example of universal instantiation.

5. Justify the rule of universal transitivity, which states that if $\forall x(P(x) \rightarrow Q(x))$ and $\forall x(Q(x) \rightarrow R(x))$ are true, then $\forall x(P(x) \rightarrow R(x))$ is true, where the domains of all quantifiers are the same.

   Solution: Take as given the statements $\forall x(P(x) \rightarrow Q(x))$ and $\forall x(Q(x) \rightarrow R(x))$. For all x such that $P(x)$, we have $Q(x)$ from the first statement, and so we have $R(x)$ from the second statement; therefore $R(x)$ is true whenever $P(x)$ is true.