Worksheet 24: April 24

Principles to Remember

Buckle up: we have a lot of terminology to cover!

- A **graph** $G = (V, E)$ consists of V, a set of **vertices** (or **nodes**), and E, a set of **edges**. Each edge has either one or two vertices associated with it, called its **endpoints**. An edge is said to **connect** its endpoints.

- In a **directed graph**, the edges (u, v) and (v, u) correspond to *different* edges between the vertices u and v, and edges are drawn as arrows. In an **undirected graph**, (u, v) and (v, u) correspond to *the same* edge between u and v, and edges are drawn as lines.

- In an undirected graph, v is a **neighbor** of u if it shares an edge with u. This edge is said to **connect** u and v, and is called **incident** with u and v. The **degree** of v, denoted as $\deg(v)$, is the number of edges incident to v, except every edge from v to itself counts double.

- In a directed graph, the **in-degree** of a vertex v is the number of edges with v as their terminal vertex, and is denoted $\deg^-(v)$. The **out-degree** of v is the number of edges with v as their initial vertex, and is denoted $\deg^+(v)$.

- Some classes of simple graphs include **cycles**, **wheels**, **path graphs**, and **complete graphs**; know them.

- In a **bipartite graph**, the vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.

- The **adjacency list** of a graph specifies which vertices are adjacent to each vertex of the graph. The **adjacency matrix** A displays the same information in a different format: if $V = \{v_1, \ldots, v_n\}$ and a_{ij} is the element A in row i and column j, then $a_{ij} = 1$ if $(v_i, v_j) \in E$, and $a_{ij} = 0$ if $(v_i, v_j) \notin E$.

- Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a bijective function $f : V_1 \to V_2$ such that

  $$(a \text{ and } b \text{ are adjacent in } G_1) \longleftrightarrow (f(a) \text{ and } f(b) \text{ are adjacent in } G_2)$$

  Think of it like being able to contort the first graph into the second.

- A **graph invariant** is a property of a graph which is preserved by isomorphism. Examples include: number of vertices; number of edges; number of vertices with degree k. Proving that two graphs are isomorphic is hard, but proving that they are *not* isomorphic can be done by finding some invariant which they don't share.

Exercises

1. Draw the following graphs:

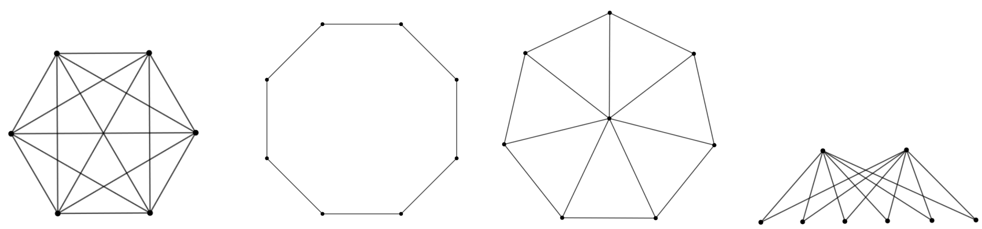
   (a) The complete graph K_6.

   (b) The cycle graph C_8.

   (c) The wheel graph W_7.

   (d) The complete bipartite graph $K_{2,6}$.

   Solution: In order:



2. How many vertices and edges do each of the above graphs have? How many vertices and edges do K_n, C_n, and W_n have for general n? How many vertices and edges does $K_{m,n}$ have for general m and n?
   Solution:

   - K_6 has 6 vertices and 21 edges; K_n has n vertices and $\binom{n}{2}$ edges.
   - C_8 has 8 vertices and 8 edges; C_n has n vertices and n edges.
   - W_7 has 8 vertices and 14 edges; W_n has $n+1$ vertices and $2n$ edges.
   - $K_{2,6}$ has 8 vertices and 12 edges; $K_{m,n}$ has $m+n$ vertices and mn edges.

3. (a) Show that K_n is not bipartite when $n \geq 3$.
   Solution: If V is the set of vertices of K_n for $n \geq 3$, then in any partition (V_1, V_2) of V, the pigeonhole principle stipulates that either V_1 or V_2 must contain at least two vertices. Because K_n is complete, these two vertices have an edge between them, and so (V_1, V_2) is not a bipartition. Since no bipartition exists, K_n is not bipartite.

   (b) Show that W_n is not bipartite when $n \geq 2$.
   Solution: Pick two adjacent vertices on the wheel and the center vertex. These three vertices are all connected to each other; there's no way to form a bipartition of them.

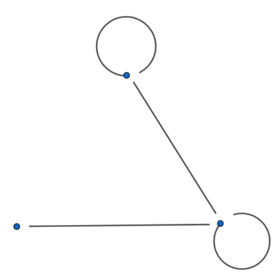
   (c) For which values of $n \geq 1$ is C_n bipartite?
   Solution: C_n is bipartite when if and only if n is even.

4. Show that the number of undirected graphs on n vertices is $2^{\binom{n}{2}}$.
Solution: There are $\binom{n}{2}$ possible edges between the n elements, and each edge can be either in the graph or not in the graph.

5. (a) Let R be a relation from A to B. Explain how we may think of R as a directed, bipartite graph.
Solution: We can form a graph $((A, B), R)$, where (A, B) is the bipartition of the vertex set $A \cup B$, and R is the set of edges. Each edge goes from an element of A to an element of B.

(b) Let R be a relation on a set A. Explain how we may think of R as a directed graph. If R is symmetric, explain how we may think of R as an undirected graph.
Solution: Similarly, we can think of A as the set of vertices and R as the set of edges. If R is symmetric, then every directed edge has a reversed counterpart in the graph; we cam delete these mirrors and remove the arrowheads.

6. Draw the graph with adjacency matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. What is the degree of each vertex?
What is the adjacency list of this graph?
Solution: The graph is drawn below.



Vertex a has degree 2. Vertex b has degree 1. Vertex c has degree 3. The adjacency list is:

Vertex	Adjacent Vertices
a	a, c
b	c
c	a, b, c

7. Show that being bipartite is a graph invariant. (Let G and H be isomorphic graphs, and suppose G is bipartite; then show that H is also bipartite.)
Solution: Let V be the vertex set of G; bipartition V as (V_1, V_2). Let f be the isomorphism from G to H. We can prove that $(f(V_1), f(V_2))$ is a bipartition of the vertex set of H. If $u, v \in f(V_1)$, then $f-1(u)$ and $f^{-1}(v)$ are both in V_1 and are thus not adjacent, so u and v are not adjacent. A similar argument holds for $u, v \in f(V_2)$. If $u \in f(V_1)$ and $v \in f(V_2)$, then $f^{-1}(u) \in V_1$ and $f^{-1}(v) \in V_2$, so $f^{-1}(u)$ and $f^{-1}(v)$ are adjacent, and thus u and v are adjacent. Therefore two vertices of H are adjacent if and only if one of them is in $f(V_1)$ and the other is in $f(V_2)$, so $(f(V_1), f(V_2))$ is a bipartition.

8. Prove that isomorphism of simple graphs is an equivalence relation.
Solution: We tackle the three properties in order.
   - Reflexive: Every graph is isomorphic to itself; the isomorphism is the identity function.
   - Symmetric: If f is an isomorphism from G_1 to G_2, then f^{-1} is an isomorphism from G_2 to G_1.
   - Transitive: If f is an isomorphism from G_1 to G_2 and h is an isomorphism from G_2 to G_3, then $f \circ g$ is an isomorphism from G_1 to G_3.

9. How many nonisomorphic simple graphs are there with
   (a) five vertices and three edges?
   Solution: 4
   (b) six vertices and four edges?
   Solution: 6