Worksheet 24: April 24

Principles to Remember

Buckle up: we have a lot of terminology to cover!

- A **graph** $G = (V, E)$ consists of V, a set of **vertices** (or **nodes**), and E, a set of **edges**. Each edge has either one or two vertices associated with it, called its **endpoints**. An edge is said to **connect** its endpoints.
- In a **directed graph**, the edges (u, v) and (v, u) correspond to *different* edges between the vertices u and v, and edges are drawn as arrows. In an **undirected graph**, (u, v) and (v, u) correspond to *the same* edge between u and v, and edges are drawn as lines.
- In an undirected graph, v is a **neighbor** of u if it shares an edge with u. This edge is said to **connect** u and v, and is called **incident** with u and v. The **degree** of v, denoted as $\deg(v)$, is the number of edges incident to v, except every edge from v to itself counts double.
- In a directed graph, the **in-degree** of a vertex v is the number of edges with v as their terminal vertex, and is denoted $\deg^-(v)$. The **out-degree** of v is the number of edges with v as their initial vertex, and is denoted $\deg^+(v)$.
- Some classes of simple graphs include **cycles**, **wheels**, **path graphs**, and **complete graphs**; know them.
- In a **bipartite graph**, the vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.
- The **adjacency list** of a graph specifies which vertices are adjacent to each vertex of the graph. The **adjacency matrix** A displays the same information in a different format: if $V = \{v_1, \ldots, v_n\}$ and a_{ij} is the element A in row i and column j, then $a_{ij} = 1$ if $(v_i, v_j) \in E$, and $a_{ij} = 0$ if $(v_i, v_j) \notin E$.
- Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a bijective function $f : V_1 \to V_2$ such that

  $$(a \text{ and } b \text{ are adjacent in } G_1) \longleftrightarrow (f(a) \text{ and } f(b) \text{ are adjacent in } G_2)$$

  Think of it like being able to contort the first graph into the second.
- A **graph invariant** is a property of a graph which is preserved by isomorphism. Examples include: number of vertices; number of edges; number of vertices with degree k. Proving that two graphs are isomorphic is hard, but proving that they are *not* isomorphic can be done by finding some invariant which they don't share.

Exercises

1. Draw the following graphs:

   (a) The complete graph K_6.

   (b) The cycle graph C_8.

   (c) The wheel graph W_7.

   (d) The complete bipartite graph $K_{2,6}$.

2. How many vertices and edges do each of the above graphs have? How many vertices and edges do K_n, C_n, and W_n have for general n? How many vertices and edges does $K_{m,n}$ have for general m and n?

3. (a) Show that K_n is not bipartite when $n \geq 3$.

   (b) Show that W_n is not bipartite when $n \geq 2$.

   (c) For which values of $n \geq 1$ is C_n bipartite?

4. Show that the number of undirected graphs on n vertices is $2^{\binom{n}{2}}$.

5. (a) Let R be a relation from A to B. Explain how we may think of R as a directed, bipartite graph.

   (b) Let R be a relation on a set A. Explain how we may think of R as a directed graph. If R is symmetric, explain how we may think of R as an undirected graph.

6. Draw the graph with adjacency matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. What is the degree of each vertex? What is the adjacency list of this graph?

7. Show that being bipartite is a graph invariant. (Let G and H be isomorphic graphs, and suppose G is bipartite; then show that H is also bipartite.)

8. Prove that isomorphism of simple graphs is an equivalence relation.

9. How many nonisomorphic simple graphs are there with

   (a) five vertices and three edges?

   (b) six vertices and four edges?