Worksheet 23: April 22 (Solutions)

Principles to Remember

- A **partial ordering** R on a set S is a relation that is reflexive, antisymmetric, and transitive. A **poset** is the pairing of a set and a partial ordering, and is denoted by (S, R).
- We usually denote partial orderings by the symbol $\preceq$. If $\preceq$ is a partial ordering and $(a, b) \in \preceq$, we write $a \preceq b$. Through an abuse of notation, if we also have $a \neq b$, we write $a \prec b$.
- Two elements of a poset $(S, \preceq)$ are **comparable** $a \preceq b$ or $b \preceq a$. If every element is comparable with every other, we say $\preceq$ is a **total relation** and call $(S, \preceq)$ a **totally ordered set** or a **chain**.
- If S is a partially ordered set, an element $a \in S$ is **minimal** if there is no smaller element in S; that is, if $b \preceq a$, then $b = a$. Similarly, a is **maximal** if there is no larger element in S; that is, if $a \preceq b$, then $a = b$.
- Relations can be represented using directed graphs: draw a vertex for each element, and for all elements (a, b) of the relation, draw an edge from a's vertex to b's vertex.
- The **Hasse diagram** for a finite poset is constructed as follows: Start with the directed graph for the relation. Erase all edges from each vertex to itself. Erase all edges which must be in the relation because of transitivity; that is, if there exists z such that $x \prec z$ and $z \prec y$, erase the edge (x, y). Finally, arrange the diagram so that x is drawn below y whenever $x \prec y$, and remove the arrowheads from the edges.

Exercises

1. Which of these relations on $\{1, 2, 3, 4\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

   (a) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
   Solution: This is a partial ordering.

   (b) $\{(1, 1), (2, 2), (3, 1), (3, 3), (3, 4), (4, 3), (4, 4)\}$
   Solution: This is not a partial ordering because it is neither antisymmetric nor transitive.

   (c) $\{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$
   Solution: This is a partial ordering.

   (d) $\{(1, 1), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
   Solution: This is a partial ordering.

(e) $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,4)\}$.
Solution: This is not a partial ordering because it is not antisymmetric.

2. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

   (a) a is taller than b?
   Solution: Not a poset, because R is not reflexive: you can't be taller than yourself.

   (b) a is not taller than b?
   Solution: Poset.

   (c) $a = b$ or a is an ancestor of b?
   Solution: Poset.

   (d) a and b have a common friend?
   Solution: Not a poset, because R is not antisymmetric.

3. Which of $\{=, \neq, <, \leq, >, \geq, |, \not|\}$ are partial orderings on $\mathbb{Z}$?
Solution: $=$, $\leq$, $\geq$, and $|$ are partial orderings. $\neq$, $<$, and $>$ are not reflexive, and $\not|$ doesn't satisfy any of the three properties of a partial ordering.

4. Prove that every partial ordering on a finite nonempty set has a minimal element.
Proof: To quote Prof. Rosen's lecture slides: "Choose an element of S. If it's not minimal, there's a smaller element. Is this second element minimal? If not, choose a smaller element. Keep going; we will stop at some point (finding then a minimal element) because the set is finite."

5. Draw the Haase diagrams for the following posets. What are the maximal and minimal elements?

   (a) $(\{0,1,2,3,4,5\}, \geq)$

   (b) $(\{1,2,3,4,5,6\}, |)$

   (c) $(\{1,2,3,6,12,24,36,48\}, |)$

   (d) $(P(\{a,b,c\}), \subseteq)$

   Solution: I still haven't learned how to use proper image editing software, so please enjoy the PowerPoint graphics on the next page.

   (a) Maximal element: 0. Minimal element: 5.

   (b) Maximal elements: 4 and 6. Minimal element: 1.

   (c) Maximal elements: 48 and 36. Minimal element: 1.

   (d) Maximal element: $\{a, b, c\}$. Minimal element: $\emptyset$.


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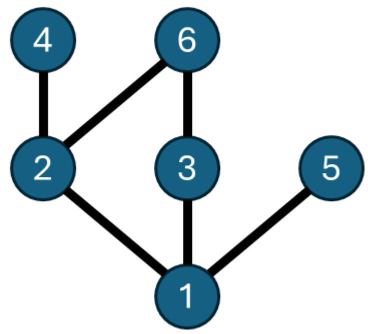
1
2
3
4
5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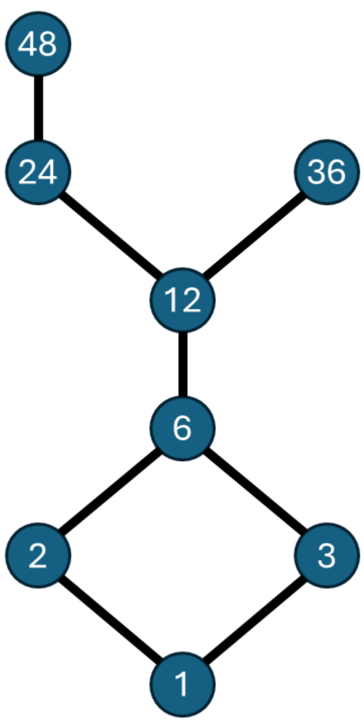
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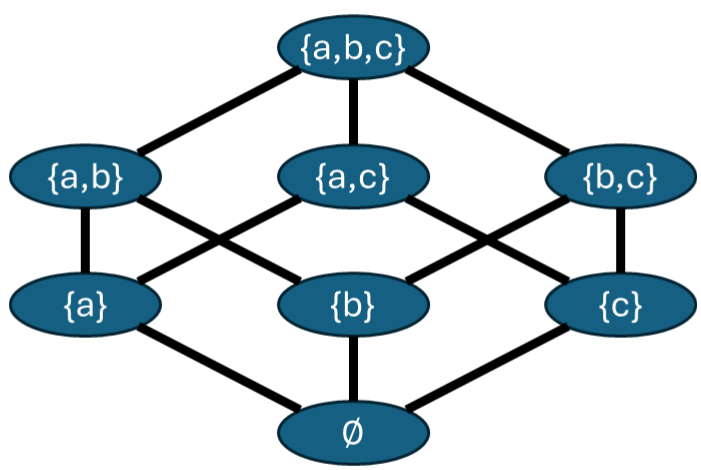
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{b}
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∅