Worksheet 23: April 22

Principles to Remember

- A **partial ordering** R on a set S is a relation that is reflexive, antisymmetric, and transitive. A **poset** is the pairing of a set and a partial ordering, and is denoted by (S, R).
- We usually denote partial orderings by the symbol $\preceq$. If $\preceq$ is a partial ordering and $(a, b) \in \preceq$, we write $a \preceq b$. Through an abuse of notation, if we also have $a \neq b$, we write $a \prec b$.
- Two elements of a poset $(S, \preceq)$ are **comparable** $a \preceq b$ or $b \preceq a$. If every element is comparable with every other, we say $\preceq$ is a **total relation** and call $(S, \preceq)$ a **totally ordered set** or a **chain**.
- If S is a partially ordered set, an element $a \in S$ is **minimal** if there is no smaller element in S; that is, if $b \preceq a$, then $b = a$. Similarly, a is **maximal** if there is no larger element in S; that is, if $a \preceq b$, then $a = b$.
- Relations can be represented using directed graphs: draw a vertex for each element, and for all elements (a, b) of the relation, draw an edge from a's vertex to b's vertex.
- The **Hasse diagram** for a finite poset is constructed as follows: Start with the directed graph for the relation. Erase all edges from each vertex to itself. Erase all edges which must be in the relation because of transitivity; that is, if there exists z such that $x \prec z$ and $z \prec y$, erase the edge (x, y). Finally, arrange the diagram so that x is drawn below y whenever $x \prec y$, and remove the arrowheads from the edges.

Exercises

1. Which of these relations on $\{1, 2, 3, 4\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

   (a) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

   (b) $\{(1, 1), (2, 2), (3, 1), (3, 3), (3, 4), (4, 3), (4, 4)\}$

   (c) $\{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$

   (d) $\{(1, 1), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

   (e) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)\}$.

2. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

   (a) a is taller than b?

   (b) a is not taller than b?

   (c) $a = b$ or a is an ancestor of b?

   (d) a and b have a common friend?

3. Which of $\{=, \neq, <, \leq, >, \geq, |, \nmid\}$ are partial orderings on $\mathbb{Z}$?

4. Prove that every partial ordering on a finite nonempty set has a minimal element.

5. Draw the Haase diagrams for the following posets. What are the maximal and minimal elements?

   (a) $(\{0, 1, 2, 3, 4, 5\}, \geq)$

   (b) $(\{1, 2, 3, 4, 5, 6\}, |)$

   (c) $(\{1, 2, 3, 6, 12, 24, 36, 48\}, |)$

   (d) $(\mathcal{P}(\{a, b, c\}), \subseteq)$