Worksheet 22: April 17 (Solutions)

Principles to Remember

- From a few units ago: A **Cartesian product** $A \times B$ of two sets A and B is defined as $\{(a, b) : a \in A, b \in B\}$. So, for example,

$$\{2, 5\} \times \{\alpha, \beta, \gamma\} = \{(2, \alpha), (2, \beta), (2, \gamma), (5, \alpha), (5, \beta), (5, \gamma)\}.$$

  Note that these are *ordered* pairs: $(2, \gamma)$ is not the same as $(\gamma, 2)$.

- A **binary relation from $\boldsymbol{A}$ to $\boldsymbol{B}$** is a subset of $A \times B$.
  - Let R be a relation. If $(a, b) \in R$, we say a is **related** to b by R.
- A **relation on a set $\boldsymbol{A}$** is a relation from A to A; that is, a subset of $A \times A$.
- Let R be a relation on a set A. We say that R is
  - **reflexive** if $(a, a) \in R$ for every element $a \in A$;
  - **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$;
  - **antisymmetric** if for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$;
  - **transitive** if for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.
- An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.
- If R is an equivalence relation on a set A, the **equivalence class** $[a]_R$ of a is the set of all elements which are related to a; that is, $[a]_R = \{s : (a, s) \in R\}$.

Exercises

1. Name some elements of each of the following relations. What properties (reflexive, symmetric, antisymmetric, transitive) do each of the relations have?

   (a) the relation on $\mathbb{Z}^+$ defined by $\{(a, b) : a \text{ divides } b\}$
   Solution: Reflexive, antisymmetric, transitive.

   (b) the relation on $\mathbb{Z}$ defined by $\{(a, b) : a \equiv b \pmod 5\}$
   Solution: Reflexive, symmetric, transitive.

   (c) the relation on $\mathbb{R}$ defined by $\{(a, b) : a < b\}$
   Solution: Antisymmetric, transitive

   (d) the relation on the set of all propositional statements defined by
   $\{(a, b) : a \to b \text{ is true}\}$
   Solution: Reflexive, transitive

(e) the relation on the set of convergent sequences of real numbers defined by $\{(a, b) : \lim a = \lim b\}$
Solution: Reflexive, symmetric, transitive

(f) the relation on Berkeley math courses defined by $\{(a, b) : a \text{ is a prereq for } b\}$
Solution: Antisymmetric, transitive

2. Which of the relations in the previous problem are equivalence relations? What are their equivalence classes?
Solution: The relations from parts (b) and (e) are equivalence relations. For part (b), the equivalence classes are the sets $\{x : x \equiv a \pmod 5\}$ for $a \in \{0, 1, 2, 3, 4\}$. For part (e), the equivalence classes are the sets $\{x : \lim a = \lim x\}$ for convergent sequences a.

3. Prove that the identity relation $R = \{(a, a) : a \in A\}$ is the only relation on A which is reflexive, symmetric, antisymmetric, and transitive.
Solution: Assume a relation T satisfies all four properties, and let $(a, b) \in T$. Then because T is symmetric, we have $(b, a) \in T$. Then because T is antisymmetric and (a, b) and (b, a) are both in T, $a = b$. Therefore elements are not related to any elements except themselves, so T is a subset of the identity relation R. Because T is reflexive, $(a, a) \in T$ for all $a \in A$, so $R \subseteq T$. Because $T \subseteq R$ and $R \subseteq T$, we have $T = R$. Therefore the identity relation is the only relation on A which satisfies the first three properties, and it's straightforward to check that it satisfies the transitive property as well.

4. What's wrong with the following "proof" that symmetric, transitive relations are reflexive?

> Let R be a symmetric, transitive relation on a set A, and let $a \in A$. Then $(a, b) \in R$, so because R is symmetric, we have $(b, a) \in R$. Then because R is transitive and $(a, b), (b, a) \in R$, we also have $(a, a) \in R$, so R is reflexive.

Point out the mistake, and give a counterexample to show that the statement is actually false.
Solution: The mistake is the statement "Then $(a, b) \in R$". b is not defined here; there is no guarantee that a relates to any element of A. As a counterexample, consider $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 3), (3, 1), (3, 3)\}$.

5. C^1 is the set of continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Let R be the relation on C^1 consisting of all pairs (f, g) such that $f'(x) = g'(x)$ for all real numbers x. Show that R is an equivalence relation. What is the equivalence class of $f(x) = x^2$?
Solution: R is reflexive because $f'(x) = f'(x)$ for all f. It is symmetric because $f'(x) = g'(x)$ if and only if $g'(x) = f'(x)$. It is transitive because if $f'(x) = g'(x)$ and $g'(X) = h'(x)$, then $f'(x) = h'(x)$. Therefore R is an equivalence relation. The equivalence class of $f(x) = x^2$ is $\{x^2 + c : c \in \mathbb{R}\}$.

6. How many relations are there on the set $\{a, b, c, d\}$? How many of them contain (a, a)?
Solution: Remember that relations are subsets of Cartesian products. There are $4^2 = 16$ elements of $\{a, b, c, d\} \times \{a, b, c, d\}$, so there are $2^{16} = 65536$ relations on $\{a, b, c, d\}$. Of these, $215 = 32768$ contain (a, a).