Worksheet 22: April 17

Principles to Remember

- From a few units ago: A **Cartesian product** $A \times B$ of two sets A and B is defined as $\{(a, b) : a \in A, b \in B\}$. So, for example,

$$\{2, 5\} \times \{\alpha, \beta, \gamma\} = \{(2, \alpha), (2, \beta), (2, \gamma), (5, \alpha), (5, \beta), (5, \gamma)\}.$$

  Note that these are *ordered* pairs: $(2, \gamma)$ is not the same as $(\gamma, 2)$.

- A **binary relation from *A* to *B*** is a subset of $A \times B$.
  - Let R be a relation. If $(a, b) \in R$, we say a is **related** to b by R.
- A **relation on a set *A*** is a relation from A to A; that is, a subset of $A \times A$.
- Let R be a relation on a set A. We say that R is
  - **reflexive** if $(a, a) \in R$ for every element $a \in A$;
  - **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$;
  - **antisymmetric** if for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$;
  - **transitive** if for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.
- An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.
- If R is an equivalence relation on a set A, the **equivalence class** $[a]_R$ of a is the set of all elements which are related to a; that is, $[a]_R = \{s : (a, s) \in R\}$.

Exercises

1. Name some elements of each of the following relations. What properties (reflexive, symmetric, antisymmetric, transitive) do each of the relations have?
   (a) the relation on $\mathbb{Z}^+$ defined by $\{(a, b) : a \text{ divides } b\}$
   (b) the relation on $\mathbb{Z}$ defined by $\{(a, b) : a \equiv b \pmod 5\}$
   (c) the relation on $\mathbb{R}$ defined by $\{(a, b) : a < b\}$
   (d) the relation on the set of all propositional statements defined by $\{(a, b) : a \to b \text{ is true}\}$
   (e) the relation on the set of convergent sequences of real numbers defined by $\{(a, b) : \lim a = \lim b\}$
   (f) the relation on Berkeley math courses defined by $\{(a, b) : a \text{ is a prereq for } b\}$

2. Which of the relations in the previous problem are equivalence relations? What are their equivalence classes?

3. Prove that the identity relation $R = \{(a, a) : a \in A\}$ is the only relation on A which is reflexive, symmetric, antisymmetric, and transitive.

4. What's wrong with the following "proof" that symmetric, transitive relations are reflexive?

   > Let R be a symmetric, transitive relation on a set A, and let $a \in A$. Then $(a, b) \in R$, so because R is symmetric, we have $(b, a) \in R$. Then because R is transitive and $(a, b), (b, a) \in R$, we also have $(a, a) \in R$, so R is reflexive.

   Point out the mistake, and give a counterexample to show that the statement is actually false.

5. C^1 is the set of continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Let R be the relation on C^1 consisting of all pairs (f, g) such that $f'(x) = g'(x)$ for all real numbers x. Show that R is an equivalence relation. What is the equivalence class of $f(x) = x^2$?

6. How many relations are there on the set $\{a, b, c, d\}$? How many of them contain (a, a)?