Worksheet 21: April 15 (Solutions)

Principles to Remember

- Consider a **recurrence relation** of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

  This relation

  - is **linear:** the right-hand side is a sum of previous terms of the sequence, all multiplied by a function of n;
  - is **homogeneous:** no terms appear which are not multiples of the a_j's;
  - has **degree *k*:** a_n is expressed in terms of the previous k terms;
  - has **constant coefficients:** the c_j's are real numbers that do not depend on n.

  Therefore, we say that this is a **linear homogeneous recurrence relation of degree *k* with constant coefficients.** (Got all that?)

- The above relation's **characteristic equation** is $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$.

- If the characteristic equation has ***distinct*** roots $r_1, r_2, \ldots, r_k$, then a sequence $\{a_n\}$ solves the recurrence relation if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

  for some $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$. (Section 8.2, Theorem 3)

- A **linear *non*homogeneous recurrence relation of degree *k* with constant coefficients** takes the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

  for some function $F(n)$ that is not identically zero.

- The **associated homogeneous recurrence relation** of the above nonhomogeneous relation is $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$; it's what you get when you take out the $F(n)$.

- Let $\{a_n^{(p)}\}$ be a particular solution of the above nonhomogeneous relation. Then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the homogeneous relation. (Section 8.2, Theorem 5)

Exercises

For the following exercises, first find the characteristic equation corresponding to the recurrence relation; then factor the polynomial; then find a general solution for the relation; then find a particular solution for the given initial terms.

1. $a_n = a_{n-1} + a_{n-2}$; $a_0 = 2$, $a_1 = 1$
   Solution: Characteristic equation: $r^2 - r - 1 = 0$. Factored: $(r - \alpha)(r - \beta) = 0$. General solution: $a_n = b\alpha^n + c\beta^n$. Particular solution: $a_n = \alpha_n + \beta^n$. These are the Lucas numbers starting at 2.

2. $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$; $a_0 = 1$, $a_1 = 1$, $a_2 = 1$
   Solution: Characteristic equation: $r^3 - 6r^2 + 11r - 6 = 0$. Factored: $(r - 1)(r - 2)(r - 3) = 0$. General solution: $a_n = \alpha_1 + \alpha_2 2^n + \alpha_3 3^n$. Particular solution: $\alpha_n = 1$.

3. $a_n = 2a_{n-1} + 3a_{n-2}$; $a_0 = 1$, $a_1 = 5$
   Solution: Characteristic equation: $r^2 - 2r - 3 = 0$. Factored: $(r - 3)(r + 1) = 0$. General solution: $a_n = \alpha_1(-1)^n + \alpha_2 3^n$. Particular solution: $a_n = \frac{-1}{2}(-1)^n + \frac{3}{2}3^n$.

4. $a_n = 7a_{n-2} + 6a_{n-3}$; $a_0 = 2$, $a_1 = 3$, $a_2 = 7$
   Solution: Characteristic equation: $r^3 - 7r - 6 = 0$. Factored: $(r + 2)(r + 1)(r - 3) = 0$. General solution: $a_n = \alpha_1(-2)^n + \alpha_2(-1)^n + \alpha_3 3^n$. Particular solution: $a_n = -(-2)^n + 2(-1)^n + 3^n$.

5. $a_n = 2a_{n-1} + 3^n$; $a_0 = 5$
   Solution: Characteristic equation: $r - 2 = 0$. Factored: $r - 2 = 0$. General solution to homogeneous relation: $a_n = \alpha_1 2^n$. Particular solution to nonhomogeneous relation: $a_n = 3^{n+1}$. General solution to nonhomogeneous relation: $a_n = 3^{n+1} + \alpha_1 2^n$. Particular solution for given starting terms: $a_n = 3^{n+1} + 2^{n+1}$.

6. $a_n = 5a_{n-1} - 6a_{n-2} + 9$; $a_0 = 15/2$, $a_1 = 25/2$
   Solution: Characteristic equation: $r^2 - 5r + 6 = 0$. Factored: $(r - 3)(r - 2) = 0$. General solution to homogeneous relation: $a_n = \alpha_1 3^n + \alpha_2 2^n$. Particular solution to nonhomogeneous relation: $a_n = \frac{9}{2}$. General solution to nonhomogeneous relation: $a_n = \alpha_1 3^n + \alpha_2 2^n + \frac{9}{2}$. Particular solution for given starting terms: $a_n = 2 \cdot 3^n + 2^n + \frac{9}{2}$.

7. $a_n = 5a_{n-1} - 6a_{n-2} + n$; $a_0 = 11/4$, $a_1 = 21/4$
   Solution: Characteristic equation: $r^2 - 5r + 6 = 0$. Factored: $(r - 3)(r - 2) = 0$. General solution to homogeneous relation: $a_n = \alpha_1 3^n + \alpha_2 2^n$. Particular solution to nonhomogeneous relation: $a_n = \frac{n}{2} + \frac{7}{4}$. General solution to nonhomogeneous relation: $a_n = \alpha_1 3^n + \alpha_2 2^n + \frac{n}{2} + \frac{7}{4}$. Particular solution for given starting terms: $a_n = 3^n + \frac{n}{2} + \frac{7}{4}$.